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Canonical forms for systems of two second-order ordinary differential equations

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Abstract

We obtain non-similar classes of realizations for real three- and four-dimensional Lie algebras in the space of vector fields in three variables. This is applied to the classification and integration of systems of two second-order ordinary differential equations (ODEs) admitting four-dimensional symmetry Lie algebras. Thus we obtain an analogue of Lie's method of integrating scalar second-order ODEs admitting two-dimensional symmetry Lie algebras for systems of two second-order ODEs. Applications to physical problems are presented.

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1. Introduction

The theory of Lie groups and Lie algebras originated from Lie's early works related to the integration of scalar ordinary differential equations (ODEs) [1]. In the study of invertible transformations leaving a given differential equation invariant, Lie noticed that those forming a one-parameter group can be described equivalently by using the vector tangent to the orbits of the group. He called this vector the *symbol* of the group and *symmetry* of the underlying equation. He showed that the set of symmetries of a differential equation forms an *infinitesimal group* (a Lie algebra in the modern terminology, see, e.g., [3, 4]), and that the integrability of the equation depends upon the properties of this *infinitesimal group*. Namely, he showed that a scalar nth-order ODE admitting an n-dimensional solvable Lie algebra of symmetries is integrable by quadratures. Since two-dimensional Lie algebras are solvable, Lie gave an algorithm for integrating second-order ODEs having two-dimensional Lie algebras of symmetries. Furthermore, he explicitly classified, in the complex domain, scalar second-order ODEs possessing symmetries. The classification in the real domain was given more recently in Mahomed and Leach [5].

Since systems of two second-order ODEs occur frequently in applications (classical, fluid and quantum mechanics, general relativity, etc), it is worth developing a classification scheme similar to Lie's for such systems. Lie [2] gave the complete classification of complex primitive Lie algebras in terms of vector fields in (1 + 2)-dimensional space: he obtained eight non-similar classes. For complex imprimitive Lie algebras, he sketched in three steps the method to follow and he implemented only the first two steps.

In this paper, we investigate realizations of real three- and four-dimensional Lie algebras in terms of vector fields in $(1+2)$-dimensional space. Furthermore, we classify all systems of two second-order ODEs admitting real four-dimensional symmetry Lie algebras and we show how one can integrate the underlying equations. Finally, we provide applications of our results to physical problems.

2. Realizations of three- and four-dimensional real Lie algebras in terms of vector fields in (1 + 2)-dimensional space

When we calculate the symmetries of a given differential equation, we find the generators explicitly in the form of vector fields (or first-order linear operators), and only afterwards do we compute the commutators to get the structure constants of the particular Lie algebra we have found. But we could also proceed backwards, that is, start from a given Lie algebra with a set of structure constants and ask which vector fields in at most three variables satisfy the given set of commutator relations with none of the vector fields vanishing. We thus ask for possible *realizations* or *representations* of our Lie algebra. Two realizations of the same Lie algebra will be considered equivalent or similar if there exists an invertible transformation mapping one of the realizations to the other.

In this section we are concerned with finding all non-similar representations of three- and four-dimensional real Lie algebras in $(1+2)$-dimensional space. We adopt the Mubarakzyanov [8] classification scheme of real low-dimensional Lie algebras reported in Patera and Winternitz [9] and we also exploit the enumeration of subalgebras of these Lie algebras given in the same paper. We shall use $\mathcal{L}$ as a place-holder for relevant Lie algebra(s) $\mathcal{L}_{i,j}^{a,b,R}$ (the jth algebra of dimension i with the superscripts a, b, if any, indicating parameters on which the algebra depends; also superscripts R, if any, indicates the algebra realizations). Moreover, following Lie, we use the shorthand notation

$$l = \frac{\partial}{\partial t} \qquad p = \frac{\partial}{\partial x} \qquad q = \frac{\partial}{\partial y}.$$

Finally, the elements of a basis of a given Lie algebra are named X_i, where i is lesser than or equal to the dimension of the underlying real Lie algebra.

2.1. Realizations of three-dimensional real Lie algebras

We now focus attention on the realizations of real three-dimensional Lie algebras. For each algebra we write down only the non-zero commutation relations.

$\mathcal{L}_{3,1}$ *(the Abelian three-dimensional Lie algebra)*

Let $r = \text{rank}[X_1, X_2, X_3]$. Then we consider the following cases.

(i) $r = 3$. One can find coordinates in which

$$X_1 = l \qquad X_2 = p \qquad X_3 = q.$$

(ii) $r = 2$. There is a coordinate system such that

$$X_1 = l \qquad X_2 = p \qquad X_3 = a(t, x, y)X_1 + b(t, x, y)X_2.$$

Since $[X_1, X_3] = 0$ and $[X_2, X_3] = 0$, $a = a(y)$, $b = b(y)$, renaming the variables, we obtain the realization

$$X_1 = p \qquad X_2 = q \qquad X_3 = b(t)p + a(t)q.$$

(iii) $r = 1$. There is a coordinate system in which

$$X_1 = p \qquad X_2 = a(t, x, y)p \qquad X_3 = c(t, x, y)p.$$

Now $[X_1, X_2] = 0$ and $[X_1, X_3] = 0$ imply that $a = a(t, y)$, $b = b(t, y)$. Furthermore, make the change of variables

$$\bar{t} = a(t, y) \qquad \bar{x} = x \qquad \bar{y} = b(t, y).$$

Thus

$$X_1 = \bar{p} \qquad X_2 = \bar{t}\bar{p} \qquad X_3 = \bar{y}\bar{p}.$$

Dropping the bars, we obtain the representation

$$X_1 = p \qquad X_2 = tp \qquad X_3 = yp.$$

$\mathcal{L}_{3,2}$*:* $[X_1, X_2] = X_2$

By assuming the connectedness and then the non-connectedness of X_1 and X_2, we arrive at the following cases.

(i) $X_1 = -yq$, $X_2 = q$. Let

$$X_3 = a(t, x, y)l + b(t, x, y)p + c(t, x, y)p.$$

Then $[X_1, X_3] = 0$ and $[X_2, X_3] = 0$ imply that $a = a(t, x)$, $b = b(t, x)$ and $c = 0$, i.e. $X_3 = a(t, x)l + b(t, x)p$. There exists a change of variables

$$\bar{t} = \bar{t}(t, x) \qquad \bar{x} = \bar{x}(t, x) \qquad \bar{y} = y$$

in which

$$X_1 = -\bar{y}\bar{q} \qquad X_2 = \bar{q} \qquad X_3 = \bar{p}.$$

Leaving out the bars, we obtain the realization

$$X_1 = -yq \qquad X_2 = q \qquad X_3 = p.$$

(ii) $X_1 = -xp - yq$, $X_2 = q$. The commutators $[X_1, X_3] = 0$ and $[X_2, X_3] = 0$ imply that $a = a(t), b = b(t)x, c = c(t)x$, i.e.

$$X_3 = a(t)l + b(t)xp + c(t)xq$$

If $a(t) \neq 0$, make the change

$$\bar{t} = \int \mathrm{d}t/a \qquad \bar{x} = \alpha(t)x \qquad \bar{y} = \beta(t)y$$

where α and β are solutions of the equations

$$a(t)\alpha' + b(t)\alpha = 0 \qquad a(t)\beta' + c(t)\beta = 0.$$

Omitting the bars, we obtain

$$X_1 = -xp - yq \qquad X_2 = cq \qquad X_3 = l$$

where c is a non-zero constant. Then replace X_2 by $(1/c)X_2$. Thus we obtain the realization

$$X_1 = -xp - yq \qquad X_2 = q \qquad X_3 = l.$$

If $a = 0$, we obtain the representation

$$X_1 = -xp - yq \qquad X_2 = q \qquad X_3 = b(t)xp + c(t)xq$$

where $(b(t), c(t)) \neq (0, 0)$.

$\mathcal{L}_{3,3}$*:* $[X_2, X_3] = X_1$ *(Weyl's algebra)*

The vector spaces $\langle X_1, X_2\rangle$ and $\langle X_1, X_3\rangle$ are two-dimensional Abelian subalgebras of $\mathcal{L}_{3,3}$. Since the operations

$$X_2 \longrightarrow X_3 \qquad X_3 \longrightarrow -X_2$$

do not affect the structure of the algebra, the following cases are relevant.

(i) $X_1 = l$, $X_2 = p$. Let $X_3 = a(t, x, y)l + b(t, x, y)p + c(t, x, y)q$. Then $[X_1, X_3] = 0, [X_2, X_3] = 0$ imply that

$$a = x + a(y) \qquad b = b(y) \qquad c = c(y).$$

If $c = 0$ and $b'(y) \neq 0$, effect the change of variables

$$\bar{t} = t \qquad \bar{x} = x + a(y) \qquad \bar{y} = b(y).$$

Dropping the bars, we arrive at the realization

$$X_1 = l \qquad X_2 = p \qquad X_3 = xl + yp.$$

If $c = 0$ and $b =$ constant, make the transformation

$$\bar{t} = t \qquad \bar{x} = x + a(y) \qquad \bar{y} = y$$

and replace X_3 by $(X_3 - \text{constant}X_1)$. Then, dropping the bars, we obtain

$$X_1 = l \qquad X_2 = p \qquad X_3 = xl.$$

If $c \neq 0$, make the change of variables

$$\bar{t} = t + \alpha(y) \qquad \bar{x} = x + \beta(y) \qquad y = \int \mathrm{d}y/c$$

where α and β are solutions of the equations

$$c(y)\alpha'(y) + a(y) - \alpha(y) = 0 \qquad c(y)\beta'(y) + b(y) = 0.$$

Without the bars we have

$$X_1 = l \qquad X_2 = p \qquad X_3 = xl + q.$$

(ii) $X_1 = p$, $X_2 = tp$. Let $X_3 = a(t,x,y)l + b(t,x,y)p + c(t,x,y)q$. Then $[X_1, X_3] = 0$, $[X_2, X_3] = 0$ yield

$$a = -1 \qquad b = b(t,y) \qquad c = c(t,y).$$

Perform the change of variables

$$\bar{t} = t \qquad \bar{x} = x + \alpha(t,y) \qquad \bar{y} = \beta(t,y)$$

where

$$\alpha_t - c(t,y)\alpha_y - b(t,y) = 0 \qquad \beta_t - c(t,y)\beta_y = 0.$$

We find

$$X_1 = p \qquad X_2 = tp \qquad X_3 = -l.$$

$\mathcal{L}_{3,4}$: $[X_1, X_3] = X_1$, $[X_2, X_3] = X_1 + X_2$

The vector space $\langle X_1, X_2\rangle$ is the only Abelian subalgebra of $\mathcal{L}_{3,4}$. Whence the cases:

(i) $X_1 = l$, $X_2 = p$. Let $X_3 = a(t,x,y)l + b(t,x,y)p + c(t,x,y)q$. Then $[X_1, X_3] = X_1$ and $[X_2, X_3] = X_1 + X_2$ imply that

$$a = t + x + a(y) \qquad b = x + b(y) \qquad c = c(y).$$

If $c = 0$, make the reduction using

$$\bar{t} = t + a(y) - b(y) \qquad \bar{x} = x + b(y).$$

Leaving out the bars, we obtain

$$X_1 = l \qquad X_2 = p \qquad X_3 = (t + x)l + xp.$$

If $c \neq 0$, invoke the change of variables

$$\bar{t} = t + \alpha(y) \qquad x = x + \beta(y) \qquad y = \int \mathrm{d}y/c$$

where α and β satisfy the following equations:

$$c(y)\alpha' - \alpha - \beta + a(y) = 0 \qquad c(y)\beta' - \alpha + b(y) = 0$$

we deduce

$$X_1 = l \qquad X_2 = p \qquad X_3 = (t + x)l + xp + q.$$

(ii) $X_1 = p$, $X_2 = tp$. Let $X_3 = a(t,x,y)l + b(t,x,y)p + c(t,x,y)q$. Then $[X_1, X_3] = X_1$ and $[X_2, X_3] = X_1 + X_2$ give

$$a = -1 \qquad b = x + b(t,y) \qquad c = c(t,y).$$

Effect the change

$$\bar{t} = t \qquad \bar{x} = x + \alpha(t,y) \qquad y = \beta(t,y)$$

where α and β satisfy the equations

$$\alpha_t - c(t,y)\alpha_y + \alpha - b(t,y) = 0 \qquad \beta_t - c(t,y)\beta_y = 0.$$

Suppressing the bars, we obtain

$$X_1 = p \qquad X_2 = tp \qquad X_3 = -l + xp.$$

$\mathcal{L}_{3,5}$: $[X_1, X_3] = X_1$, $[X_2, X_3] = X_2$

The vector space $\langle X_1, X_2\rangle$ is the only two-dimensional Abelian subalgebra of $\mathcal{L}_{3,5}$. Using the same method as before, we obtain the following representations:

$$
\begin{array}{lll}
X_1 = l & X_2 = p & X_3 = tl + xp \\
X_1 = l & X_2 = p & X_3 = tl + xp + q \\
X_1 = p & X_2 = tp & X_3 = xp \\
X_1 = p & X_2 = tp & X_3 = xp + q.
\end{array}
$$

$\mathcal{L}^a_{3,6}$, $a \in [-1, 1)$: $[X_1, X_3] = X_1$, $[X_2, X_3] = aX_2$

After the same kind of reasoning as before, we obtain the realizations:

$$
\begin{array}{lll}
X_1 = l & X_2 = p & X_3 = tl + axp \\
X_1 = l & X_2 = p & X_3 = tl + axp + q \\
X_1 = p & X_2 = tp & X_3 = t(1-a)l + xp.
\end{array}
$$

$\mathcal{L}^a_{3,7}$, $a \geqslant 0$: $[X_1, X_3] = aX_1 - X_2$, $[X_2, X_3] = X_1 + aX_2$

We obtain the following representations:

$$
\begin{array}{lll}
X_1 = l & X_2 = p & X_3 = (at + x)l - (t - ax)p \\
X_1 = l & X_2 = p & X_3 = (at + x)l - (t - ax)p + q \\
X_1 = p & X_2 = tp & X_3 = -(t^2 + 1)l + (a - t)xp.
\end{array}
$$

$\mathcal{L}_{3,8}$: *the vector space* $[X_1, X_2] = X_1$, $[X_2, X_3] = X_3$, $[X_3, X_1] = 2X_1$

$\langle X_1, X_2\rangle$ is the only two-dimensional subalgebra of $\mathcal{L}_{3,8}$. By assuming the connectedness and then the non-connectedness of X_1 and X_2, we arrive at the following cases:

(i) $X_1 = q$, $X_2 = yq$. Suppose $X_3 = a(t, x, y)l + b(t, x, y)p + c(t, x, y)q$. Then $[X_2, X_3] = X_3$ and $[X_3, X_1] = 2X_1$ result in

$$a = 0 \qquad b = 0 \qquad c = -y^2.$$

Whence the realization

$$X_1 = q \qquad X_2 = yq \qquad X_3 = -y^2 q.$$

(ii) $X_1 = q$, $X_2 = xp + yq$. Let $X_3 = a(t, x, y)l + b(t, x, y)p + c(t, x, y)q$. Then $[X_2, X_3] = X_3$ and $[X_3, X_1] = 2X_1$ imply that

$$a = a(t)x \qquad b = b(t)x^2 - 2xy \qquad c = c(t)x^2 - y^2.$$

Hence the representation

$$X_1 = q \qquad X_2 = yq \qquad X_3 = a(t)xl + \left(b(t)x^2 - 2xy\right) p + \left(c(t)x^2 - y^2\right) q$$

where a, b, c are arbitrary functions. By considering the cases $a \neq 0$ and $a = 0$, we deduce after suitable changes of variables, the following realizations:

$$\begin{array}{lll} X_1 = l & X_2 = tl + xp & X_3 = -t^2 l - 2xtp + xq \\ X_1 = l + p & X_2 = tl + xp & X_3 = -t^2 l - x^2 p \\ X_1 = -tp & X_2 = \frac{1}{2}(-tl + xp) & X_3 = -xl. \end{array}$$

$\mathcal{L}_{3,9}$: $[X_1, X_2] = X_3, [X_3, X_1] = X_2, [X_2, X_3] = X_1$

Note that if X_1, X_2, X_3 are connected, there is no real representation of $\mathcal{L}_{3,9}$. Hereafter, we assume that X_1, X_2 and X_3 are not connected. There is a change of variables in which $X_1 = l$. Now let

$$X_2 = a(t, x, y)l + b(t, x, y)p + c(t, x, y)q$$

and

$$X_3 = \alpha(t, x, y)l + \beta(t, x, y)p + \gamma(t, x, y)q.$$

Then $[X_1, X_2] = X_3, [X_1, X_2] = X_2, [X_2, X_3] = X_1$ give rise to

$$\begin{array}{ll} a = A_1(x, y)\cos t + A_2(x, y)\sin t & \alpha = A_2(x, y)\cos t - A_1(x, y)\sin t \\ b = B_1(x, y)\cos t + B_2(x, y)\sin t & \beta = B_2(x, y)\cos t - B_1(x, y)\sin t \\ c = A_1(x, y)\cos t + A_2(x, y)\sin t & \gamma = C_2(x, y)\cos t - C_1(x, y)\sin t \end{array}$$

where $A_i, B_i, C_i; i = 1, 2$ satisfy the system

$$\begin{aligned} B_1 A_{2,x} - B_2 A_{1,x} + C_1 A_{2,y} - C_2 A_{1,y} &= 1 + A_1^2 + A_2^2 \\ B_1 B_{2,x} - B_2 B_{1,x} + C_1 B_{2,y} - C_2 B_{1,y} &= A_1 B_1 + A_2 B_2 \\ C_1 C_{2,y} - C_2 C_{1,y} + B_1 C_{2,y} - B_2 C_{1,y} &= 1 + A_1 C_1 + A_2 C_2. \end{aligned} \tag{1}$$

Consider the change

$$\bar{x} = \bar{x}(x, y) \qquad \bar{y} = \bar{y}(x, y) \qquad \bar{t} = t + \lambda(x, y).$$

In this coordinate system,

$$\begin{aligned} X_1 &= \bar{l} \\ X_2 &= (\bar{A}_1 \cos\bar{t} + \bar{A}_2 \sin\bar{t})\bar{l} + (\bar{B}_1 \cos\bar{t} + \bar{B}_2 \sin\bar{t})\bar{p} + (\bar{C}_1 \cos\bar{t} + \bar{C}_2 \sin\bar{t})\bar{q} \\ X_3 &= (\bar{A}_2 \cos\bar{t} - \bar{A}_1 \sin\bar{t})\bar{l} + (\bar{B}_2 \cos\bar{t} - \bar{B}_1 \sin\bar{t})\bar{p} + (\bar{C}_2 \cos\bar{t} - \bar{C}_1 \sin\bar{t})\bar{q} \end{aligned}$$

where

$$\begin{aligned} \bar{A}_1 &= A_1 \cos\lambda - A_2 \sin\lambda + \lambda_x(B_1 \cos\lambda - B_2 \sin\lambda + \lambda_y(C_1 \cos\lambda - C_2 \sin\lambda) \\ \bar{A}_2 &= A_1 \sin\lambda + A_2 \cos\lambda + \lambda_x(B_1 \sin\lambda + B_2 \cos\lambda + \lambda_y(C_1 \sin\lambda + C_2 \cos\lambda) \\ \bar{B}_1 &= \bar{x}_x(B_1 \cos\lambda - B_2 \sin\lambda) + \bar{x}_y(C_1 \cos\lambda - C_2 \sin\lambda) \\ \bar{B}_2 &= \bar{x}_x(B_1 \sin\lambda + B_2 \cos\lambda) + \bar{x}_y(C_1 \sin\lambda + C_2 \cos\lambda) \\ \bar{C}_1 &= \bar{y}_x(B_1 \cos\lambda - B_2 \sin\lambda) + \bar{y}_y(C_1 \cos\lambda - C_2 \sin\lambda) \\ \bar{C}_2 &= \bar{y}_x(B_1 \sin\lambda + B_2 \cos\lambda) + \bar{y}_y(C_1 \sin\lambda + C_2 \cos\lambda). \end{aligned}$$

Now we show that we can assume that $B_2 = C_1 = 0$.

If $B_1C_2 - B_2C_1 \neq 0$, by choosing $\bar{x}$ and $\bar{y}$ as solutions of the following equations:

$$\bar{x}_x(B_1 \sin\lambda + B_2 \cos\lambda) + \bar{x}_y(C_1 \sin\lambda + C_2 \cos\lambda) = 0$$

$$\bar{y}_x(B_1 \cos\lambda - B_2 \sin\lambda) + \bar{y}_y(C_1 \cos\lambda - C_2 \sin\lambda) = 0$$

we have that $\bar{B}_2 = 0 = \bar{C}_1$.

If $B_1C_2 - B_2C_1 = 0$, then for $(B_1, B_2) = (0, 0)$, we choose $\bar{x} = x$, $\bar{y} = y$ and λ is a solution of

$$C_1 \cos\lambda + C_2 \sin\lambda = 0.$$

Note that in this case $(C_1, C_2) \neq (0, 0)$ otherwise the operators would be connected and this is excluded. Now if $(B_1, B_2) \neq (0, 0)$, we choose $\bar{x} = x$, λ a solution of

$$B_1 \sin\lambda + B_2 \cos\lambda = 0$$

and $\bar{y}$ a solution of

$$\bar{y}_x(B_1 \cos\lambda - B_2 \sin\lambda) + \bar{y}_y(C_1 \cos\lambda - C_2 \sin\lambda) = 0.$$

Henceforth we assume that $B_2 = C_1 = 0$. Thus the system (1) reduces to

$$B_1A_{2,x} - C_2A_{1,y} = 1 + A_1^2 + A_2^2$$

$$-C_2B_{1,y} = A_1B_1$$

$$B_1C_{2,y} = A_2C_2.$$

If $B_1 \neq 0$, make the change of variables

$$\bar{t} = t + \pi/4 \qquad \bar{x} = x \qquad \bar{y} = \bar{y}(x, y)$$

where $\bar{y}$ is a solution of $B_1\bar{y}_x + C_2\bar{y}_y = 0$. Hence $\bar{C}_2 = 0$. Thus we can assume that $B_1 = 0$ or $C_2 = 0$. Both cases lead to realizations equivalent to the following one:

$$X_1 = l \qquad X_2 = x\cos t l - (1 + x^2)\sin t p \qquad X_3 = -x \sin t l - (1 + x^2)\cos t p.$$

Now perform the transformation

$$\bar{t} = \tan t \qquad \bar{x} = \frac{x}{\cos t} \qquad \bar{y} = y.$$

Dropping the bars, we obtain the representation

$$X_1 = (1 + t^2)l + xtp \qquad X_2 = xl - tp \qquad X_3 = -xtl - (1 + x^2)p.$$

It is worthwhile to note that the same realization was obtained in Mahomed and Leach [5], where the authors were searching for representations in $(1+1)$-dimensional space. The results obtained are summarized in table 1.

2.2. Realizations of four-dimensional real Lie algebras

According to the classification of Mubarakzyanov [8], there are 30 real four-dimensional Lie algebras. As far as representation, and to some extent ODEs, are concerned, some of these algebras (namely those depending on parameters) may be treated simultaneously. All four-dimensional real Lie algebras contain three-dimensional subalgebras. Since we already have represented real three-dimensional Lie algebras in the previous subsection, we can build representations of four-dimensional Lie algebras on them. We shall not present details of calculations for all the cases. Nevertheless, we shall explicitly work out representations for few cases and all the representations will be summarized in table 2.

$\mathcal{L}_{4,1}$: the four-dimensional Abelian Lie algebra

The algebra $\mathcal{L}_{4,1}$ contains $\mathcal{L}_{3,1}$. Since permuting X_1, X_2, X_3 and X_4 does not affect the structure of $\mathcal{L}_{4,1}$, the following cases are to be distinguished.

(i) $X_1 = l, X_2 = p, X_3 = q$. Let $X_4 = a(t,x,y)l + b(t,x,y)p + c(t,x,y)q$. Then $[X_i, X_4] = 0, i = 1,2,3$ imply that $a =$ constant, $b =$ constant, $c =$ constant. This contradicts the fact that X_1, X_2, X_3 and X_4 form a basis of $\mathcal{L}_{4,1}$.

(ii) $X_1 = p, X_2 = q, X_3 = f(t)p + g(t)q$. The relations $[X_1, X_4] = 0$ and $[X_2, X_4] = 0$ imply that $a = a(t), b = b(t), c = c(t)$. Now $[X_3, X_4] = 0$ implies that $a(t)f'(t) = 0, a(t)g'(t) = 0$. If $a \neq 0$ then $f =$ constant and $g =$ constant. However, this would imply that X_1, X_2 and X_3 are linearly dependent. Thus $a = 0$. Whence the realization

$$X_1 = p \qquad X_2 = q \qquad X_3 = f(t)p + g(t)q \qquad X_4 = b(t)p + c(t)q$$

where $f'(t)c'(t) - g'(t)b'(t) \neq 0$ to ensure that X_1, X_2, X_3, X_4 are linearly independent.

(iii) $X_1 = p, X_2 = tp, X_3 = yp$. $[X_i, X_4] = 0, i = 1,2,3$ imply that $a = 0, b = (t,y), c = 0$. Hence the realization

$$X_1 = p \qquad X_2 = tp \qquad X_3 = yp \qquad X_4 = b(t,y)p$$

where $b \neq \text{constant} \times t + \text{constant} \times y + \text{constant}$.

The interested reader is referred to Wafo Soh and Mahomed [10] for another approach to this case.

$\mathcal{L}_{4,2}$: $[X_1, X_2] = X_2$

The algebra $\mathcal{L}_{4,2}$ contains $\mathcal{L}_{3,1}$ with basis $\{X_1, X_3, X_4\}$ or $\{X_2, X_3, X_4\}$. Hence we must consider the following cases.

(i) $X_1 = l, X_3 = p, X_4 = q$. Let $X_2 = a(t,x,y)l + b(t,x,y)p + c(t,x,y)q$. Then $[X_2, X_3] = 0 = [X_2, X_4]$ and $[X_1, X_2] = X_2$ imply that $a = a_0\mathrm{e}^t, b = b_0\mathrm{e}^t, c = c_0\mathrm{e}^t$, where a_0, b_0, c_0 are constants. Now perform the change of variables

$$\bar{t} = \mathrm{e}^t \qquad \bar{x} = x \qquad \bar{y} = y.$$

Omitting the bars, we obtain

$$X_1 = tl \qquad X_2 = a_0t^2l + b_0t + c_0tq \qquad X_3 = p \qquad X_4 = q.$$

This realization may be simplified further by considering the cases $(b_0, c_0) = 0, 0$ and $(b_0, c_0) \neq 0, 0$.

(ii) $X_1 = p, X_3 = q, X_4 = f(t)p + g(t)q$. Here $[X_2, X_3] = 0$ and $[X_1, X_2] = X_2$ imply that $a = a(t)\,\mathrm{e}^x, b = b(t)\,\mathrm{e}^x, c = c(t)\,\mathrm{e}^x$. Also $[X_2, X_4] = 0$ gives rise to

$$a(t)f(t) = 0$$
$$a(t)f'(t) - f(t)b(t) = 0$$
$$a(t)g'(t) - f(t)c(t) = 0.$$

If $a = 0$, then $f = 0$. If $a \neq 0$ then $f = 0$ and $g =$ constant. But this is inconsistent with the fact that X_3 and X_4 are independent. Hence

$$X_1 = p \qquad X_2 = b(t)\,\mathrm{e}^x p + c(t)\,\mathrm{e}^x q \qquad X_3 = q \qquad X_4 = g(t)q$$

where $g'(t) \neq 0$ since X_3 and X_4 must be linearly independent. Now, perform the transformation

$$\bar{t} = g(t) \qquad \bar{x} = x \qquad \bar{y} = y.$$

Therefore, we deduce the realization

$$X_1 = p \qquad X_2 = \eta(t)\,\mathrm{e}^x p + \mu(t)\,\mathrm{e}^x q \qquad X_3 = q \qquad X_4 = tq.$$

(iii) $X_1 = p, X_2 = tp, X_4 = yp$. This case leads to an inconsistency.

(iv) $X_2 = l, X_3 = p, X_4 = q$. Let $X_4 = a(t,x,y)l + b(t,x,y)p + c(t,x,y)q$. Then $[X_1, X_3] = 0 = [X_1, X_4]$ imply that $a = a(t), b = b(t), c = c(t)$. Now $[X_1, X_2] = X_2$ implies $a = -t + a_0, b = b_0, c = c_0$. Replacing X_1 by $X_1 - a_0X_2 - b_0X_3 - c_0X_4$, we obtain the realization

$$X_1 = -tl \qquad X_2 = l \qquad X_3 = p \qquad X_4 = q.$$

(v) $X_2 = p, X_3 = q, X_4 = f(t)p + g(t)q$. $[X_1, X_3] = 0$ and $[X_1, X_2]$ imply that $a = a(t), b = -x + b(t), c = c(t)$. The relation $[X_1, X_4] = 0$ constrains a, f, g to

$$a(t)f'(t) + f(t) = 0$$
$$a(t)g'(t) = 0.$$

If $a = 0$ then $f = 0$. Make the change of variables

$$\bar{t} = t \qquad \bar{x} = -x + b(t) \qquad \bar{y} = y.$$

Omitting the bars, we obtain the realization

$$X_1 = -xp + \mu(t)q \qquad X_2 = p \qquad X_3 = q \qquad X_4 = tq.$$

If $a \neq 0$, a suitable change of variables ($\bar{t} = \int \mathrm{d}t/a, \bar{x} = x, \bar{y} = y$) leads to $a = 1$. Hence $f = f_0\mathrm{e}^{-t}, g = g_0$, where $f_0 \neq 0$. Perform the change

$$\bar{t} = \mathrm{e}^{-t} \qquad \bar{x} = x + \alpha(t) \qquad \bar{y} = y + \beta(t)$$

where α satisfies $\alpha'(t) + \alpha(t) + b(t) = 0$ and β satisfies $\beta'(t) + c(t) = 0$. Then replace X_4 by $(X_4 - g_0X_3)/f_0$. Dropping the bars, we find

$$X_1 = -tl - xp \qquad X_2 = p \qquad X_3 = q \qquad X_4 = tp.$$

(vi) $X_2 = p, X_3 = tp, X_4 = yp$. We have that $[X_1, X_3] = 0 = [X_1, X_4]$ and $[X_1, X_2] = X_2$ imply that $a = -t, b = -x + b(t,y), c = -y$. Now, invoke the transformation

$$\bar{t} = t \qquad \bar{x} = x + \alpha(t,y) \qquad \bar{y} = y$$

where α satisfies $-t\alpha_t - y\alpha_y + \alpha + b = 0$. Dropping the bars, we obtain the realization

$$X_1 = -tl - xp - yq \qquad X_2 = p \qquad X_3 = tp \qquad X_4 = yp.$$

$\mathcal{L}_{4,3}$: $[X_1, X_2] = X_2$, $[X_3, X_4] = X_4$

Permuting the pairs (X_1, X_2) and (X_3, X_4) does not affect the structure of $\mathcal{L}_{4,3}$. Whence the following cases:

(i) $X_1 = -yq$, $X_2 = q$. Let $X_3 = a(t, x, y)l + b(t, x, y)p + c(t, x, y)q$. Then $[X_2, X_3] = 0$ and $[X_1, X_3]$ imply that $X_1 = a(t, x)l + b(t, x)p$. Similarly, $X_4 = \xi(t, x)l + \eta(t, x)p$. Thus X_3 and X_4 depend only on t and x. Since $[X_3, X_4] = X_4$, we deduce that $X_3 = -xp$ and $X_4 = p$ or $X_3 = -tl - xp$ and $X_4 = p$. Whence the representations

$$X_1 = -yq \qquad X_3 = q \qquad X_3 = -xp \qquad X_4 = p$$

and

$$X_1 = -yq \qquad X_2 = q \qquad X_3 = -tl - xp \qquad X_4 = p.$$

(ii) $X_1 = -xp - yq, X_2 = q$. Here $[X_1, X_3] = 0$ and $[X_2, X_3] = 0$ imply that $a = a(t), b = b(t)x, c = c(t)x$. Hence $X_3 = a(t)l + b(t)xp + c(t)xq$. Similarly, $X_4 = \xi(t)l + \eta(t)xp + \mu(t)xq$ and $[X_3, X_4] = X_4$ imply

$$a\xi' - a'\xi = \xi$$

$$a\eta' - a'\xi = \eta$$

$$a\mu' + b\mu - \xi c' - \eta c = \mu.$$

If $a = 0$, then $\xi = 0$ and $\eta = 0$. Therefore, $X_3 = xp + c(t)xq$, $X_4 = \mu(t)xq$. Call on the transformation

$$\bar{t} = t \qquad \bar{x} = \mu(t)x \qquad \bar{y} = y - c(t)x.$$

We find

$$X_1 = -xp - yq \qquad X_2 = q \qquad X_3 = xp \qquad X_4 = xq.$$

If $a \neq 0$, then perform the change of variables

$$\bar{t} = \int \mathrm{d}t/a \qquad \bar{x} = \alpha(t)x \qquad \bar{y} = y + \beta(t)x$$

where α satisfies $a\alpha' + b\alpha = 0$ and β satisfies $a\beta + b\beta + c = 0$. We may assume that $a = 1$ and $b = 0 = c$. Hence $\xi = \xi_0 \mathrm{e}^t$, $\eta = \eta_0 \mathrm{e}^t$, $\mu = \mu_0 \mathrm{e}^t$. Whence the realization

$$X_1 = -xp - yq \qquad X_2 = q \qquad X_3 = l \qquad X_4 = \mathrm{e}^t(\xi_0 l + \eta_0 xp + \mu_0 xq)$$

where ξ_0, η_0, μ_0 are constants. This realization can be further simplified by considering the cases $(\eta_0, \mu_0) = 0, 0$ and $(\eta_0, \mu_0) \neq 0, 0$.

We proceed in the same way as above for the remaining four-dimensional Lie algebras. The results are summarized in table 2.

3. Canonical forms and reduction of systems of two second-order ODEs having four-dimensional symmetry Lie algebras

The Lie algorithm for calculating the symmetry vectors of a given differential equation is well known (see, e.g., [6, 7, 17]). Now assume that the symmetry vectors (of some unknown equation) are given without recourse to that equation. Can one recover the equation? This question can be understood as a sort of inverse problem in symmetry analysis and is sometimes referred to as *group-theoretic modelling*.

In this section we aim at classifying systems of two second-order ODEs admitting exactly or maximally real four-dimensional symmetry Lie algebras. We also discuss the integrability of such systems.

3.1. Canonical forms

Here we only present the details of calculations for one case. The other cases can be dealt with in a similar manner. Let us begin with the following theorem which was proved in Wafo Soh and Mahomed [10].

Theorem 1. *A system of two second-order ODEs is linearizable via a point transformation if and only if it admits* $\mathcal{L}_{4,1}$ *or* $\mathcal{L}_{4,15}^{1,1}$.

Consider the algebra $\mathcal{L}_{4,2}^{1}$:

$$X_1 = -t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \qquad X_2 = \frac{\partial}{\partial x} \qquad X_3 = t\frac{\partial}{\partial x} \qquad X_4 = y\frac{\partial}{\partial x}.$$

Let us also investigate a system admitting $\mathcal{L}_{4,2}^{1}$:

$$\begin{aligned} \ddot{x} &= f(t, x, y, \dot{x}, \dot{y}) \\ \ddot{y} &= g(t, x, y, \dot{x}, \dot{y}). \end{aligned} \tag{2}$$

Following the Lie algorithm, (2) is invariant under X_2 if and only if

$$X_2^{[2]}(\ddot{x} - f)|_{(2)} = 0 \qquad X_2^{[2]}(\ddot{y} - g)|_{(2)} = 0$$

i.e.

$$f_x = 0, g_x = 0.$$

Hence $f = f(t, y, \dot{x}, \dot{y})$, $g = g(t, y, \dot{x}, \dot{y})$ and equation (2) becomes

$$\begin{aligned} \ddot{x} &= f(t, y, \dot{x}, \dot{y}) \\ \ddot{y} &= g(t, y, \dot{x}, \dot{y}). \end{aligned} \tag{3}$$

Equation (3) is invariant under X_3 if and only if

$$X_3^{[2]}(\ddot{x} - f)|_{(3)} = 0 \qquad X_3^{[2]}(\ddot{y} - g)|_{(3)} = 0$$

and

$$f_{\dot{x}} = 0 \qquad g_{\dot{x}} = 0.$$

Hence $f = f(t, y, \dot{y})$, $g = g(t, y, \dot{y})$ and equation (3) becomes

$$\begin{aligned} \ddot{x} &= f(t, y, \dot{y}) \\ \ddot{y} &= g(t, y, \dot{y}). \end{aligned} \tag{4}$$

In like manner, invariance under X_4 leads to $g = 0$ and we obtain the system

$$\begin{aligned} \ddot{x} &= f(t, y, \dot{y}) \\ \ddot{y} &= 0. \end{aligned} \tag{5}$$

Finally, invariance under X_1 requires that

$$f = -f_t - y f_y$$

i.e. $f = t^{-1}F(y/t, \dot{y})$. Thus the system which admits $\mathcal{L}^1_{4,2}$ is

$$\begin{aligned} \ddot{x} &= t^{-1}f(y/t, \dot{y}) \\ \ddot{y} &= 0. \end{aligned} \tag{6}$$

Other cases are treated similarly and the results are listed in table 3. Note that certain algebras or realizations do not appear in this table, the reason for this being simply that they are either not admitted by any equation ($\mathcal{L}_{4,10}$ for instance) or the equations which admit them have more than four symmetries ($\mathcal{L}^8_{4,19}$, $\mathcal{L}^{0,1}_{4,20}$ for example).

3.2. Integrability

Assume that we want to integrate or reduce a system of two second-order ODEs admitting a four-dimensional symmetry Lie algebra. How does the knowledge of the symmetries help us in the solution of this problem? If we assume that the four-dimensional Lie algebra in question is solvable, then we can try successive reduction as in the case of scalar equations. However, we immediately face a major problem: a vector field in three variables has a basis of first-order invariants formed by four elements. Hence in performing a reduction in the order there will be ambiguity in the choice of the new variables as invariants. Indeed, there are four possibilities. This fact emphasizes a difference between scalar and systems of ODEs. In order to avoid the situation we have mentioned, we proceed as follows. First, we reduce the system to one of the canonical forms given in table 3. If we can solve the system in its canonical form, we proceed backwards to recover the solution of the initial system. Note that the transformation bringing the system to its canonical form is simply the transformation which maps its symmetry Lie algebra to one of the realizations listed in table 2. Thus the problem of integrating systems admitting four-dimensional Lie algebras is reduced to that of integrating canonical forms. The following result can be stated by analysing the canonical forms obtained.

Proposition 1. *If a system of two ODEs admits maximally a four-dimensional symmetry Lie algebra, then this algebra has one or two functionally independent first-order differential invariants. Furthermore, the underlying equation, in terms of invariants, can either be integrated by quadratures or its integration depends on that of a first-order scalar ODE.*

As illustration, we deal with two examples. Consider, for instance, the system

$$\ddot{x} = \dot{x}^2 f(\dot{x}/\dot{y}) \qquad \ddot{y} = \dot{x}^2 g(\dot{x}/\dot{y}) \tag{7}$$

which admits $\mathcal{L}^2_{4,2}$ with basis

$$X_1 = -tl \qquad X_2 = l \qquad X_3 = p \qquad X_4 = q.$$

Note that $\mathcal{L}^1_{3,1} = \langle X_2, X_3, X_4\rangle$ is a subalgebra of $\mathcal{L}^2_{4,2}$. The first-order differential invariants of $\mathcal{L}^1_{3,1}$ are

$$u = \dot{x} \qquad v = \dot{y}.$$

In the variables t, u, v, the system becomes

$$\dot{u} = u^2 f(u/v) \qquad \dot{v} = u^2 g(u/v). \tag{8}$$

This system inherits the symmetries

$$X_1 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \qquad X_2 = \frac{\partial}{\partial t}.$$

Also note that

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{f(u/v)}{g(u/v)}$$

and u/v are invariants of X_1. This suggests the change of variable $w = u/v$:

$$\frac{\mathrm{d}w}{\mathrm{d}v} = \frac{1}{v}\left(\frac{f(w)}{g(w)} - w\right).$$

Finally, we have reduced the initial system to

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= v^2 w^2 g(w)\\ \frac{\mathrm{d}w}{\mathrm{d}v} &= \frac{1}{v}\left(\frac{f(w)}{g(w)} - w\right).\end{aligned} \tag{9}$$

This system is obviously solvable by quadratures. This example should not mislead one into believing that any system admitting a four-dimensional Lie algebra is solvable by quadratures. Let us next investigate a case where the integration of the system depends on that of a first-order scalar ODE. Consider the system

$$\ddot{x} = x f(t, \dot{x}/x) \tag{10a}$$

$$\ddot{y} = x\dot{y} f(t, \dot{x}/x) \tag{10b}$$

which admits $\mathcal{L}^2_{4,3}$. In (10a), make the change of variable

$$u = \dot{x}/x.$$

It becomes

$$\dot{u} = f(t, u) - u^2. \tag{11}$$

The integration of (10) obviously depends on that of (11).

4. Applications

First, we present examples from applications of systems of two second-order ODEs that admit point symmetry algebras. Next, we give an application in relativity, i.e. how realizations of vector fields in three variables are useful in the classification of group of motions.

1. The Hénon–Heiles problem has been posed as a model for the motion of a galactic cluster and has the Hamiltonian [11]

$$H = \tfrac{1}{2}(p_1^2 + p_2{}^2 + x^2 + y^2) + x^2 y - \tfrac{1}{3}y^2$$

($p_1 = \dot{x},\ p_2 = \dot{y}$). The Newtonian equations of motion are

$$\ddot{x} + x + 2xy = 0$$

$$\ddot{y} + y + x^2 - \tfrac{2}{3}y = 0.$$

This system only has time-translation symmetry $\partial/\partial t$ [11]. It has a further first integral besides the energy integral [11].

2. The Newtonian system [12]

$$\ddot{x} + \partial V/\partial x = 0 \qquad \ddot{y} + \partial V/\partial y = 0$$

with potential $V = \lambda_1 \ln x + \lambda_2 \ln y$, λ_i constants, has the two-dimensional non-Abelian Lie algebra generated by

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

and it is not in general completely integrable.

3. (a) The classical Kepler problem in reduced Cartesian coordinates is described by the vector equation of motion

$$\underline{\ddot{r}} + \frac{\mu}{r^3}\underline{r} = \underline{0}$$

where μ is a positive constant. As the orbits lie in a plane, in terms of polar coordinates the equation of motion can be split up into its radial and angular components as

$$\ddot{r} - r\dot{\theta}^2 + \mu r^{-2} = 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

It is well known (see, e.g., [13]) that this system has the symmetries

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial \theta} \qquad X_3 = t\frac{\partial}{\partial t} + \frac{2}{3}r\frac{\partial}{\partial r}.$$

These symmetries correspond to the constants of the motion: energy, angular momentum and the Laplace–Runge–Lenz vector [13]. The Kepler problem is completely integrable.

(b) The generalized Ermakov system in two dimensions (see, e.g., [14])

$$\ddot{x} = \frac{1}{x^3}f\left(\frac{y}{x}\right) \qquad \ddot{y} = \frac{1}{y^3}g\left(\frac{y}{x}\right)$$

where f and g are arbitrary functions of their arguments, has three symmetries

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad X_3 = t^2\frac{\partial}{\partial t} + t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

which constitute the symmetry algebra $sl(3, R)$ [14]. Furthermore, the Hamiltonian structure and integrability are discussed in [14].

4. The two-dimensional central force problem

$$\underline{\ddot{r}} + \left(\frac{\mu}{r^4} - \epsilon\right)\underline{r} = \underline{0} \qquad \mu, \epsilon > 0$$

in terms of polar coordinates is

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^3} - \epsilon r = 0 \qquad \mu, \epsilon > 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

This system has the symmetries [15]

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial \theta} \qquad X_{3,4} = \exp(\pm 2t\sqrt{\epsilon})\left(\epsilon^{-1/2}\frac{\partial}{\partial t} \pm r\frac{\partial}{\partial r}\right)$$

where the + and − in ± refer to X_3 and X_4, respectively. The integrability is discussed in [15].

In addition to the above examples, systems of two second-order ODEs arise in the symmetry reduction of systems of partial differential equations (see, for example, [6, 7]).

Notwithstanding, the realizations obtained in this paper have important applications in the classification of group of motions: given a metric g, a vector X is a Killing vector if the Lie derivative of g is zero, namely

$$\mathcal{L}_X g = 0. \tag{12}$$

The Killing vectors of a non-degenerate metric form a Lie algebra. So if one knows *a priori* a realization of a given Lie algebra, for example a four-dimensional Lie algebra, one can invoke (12) to obtain the corresponding metric. Petrov [16] utilized this approach in the classification of gravitational fields admitting simply transitive and intransitive four-dimensional group of motions. For example [16], to the realization

$$X_1 = \frac{\partial}{\partial x^1} \qquad X_2 = \frac{\partial}{\partial x^3} \qquad X_3 = \frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^3} \qquad X_4 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$$

corresponds the metric

$$\mathrm{d}s^2 = 2\,\mathrm{d}x^1\,\mathrm{d}x^4 + 2\alpha(x^4)\exp(-x^1)\,\mathrm{d}x^2\,\mathrm{d}x^3$$

where α is an arbitrary function.

5. Conclusion

We have obtained non-similar realizations of three- and four-dimensional real Lie algebras in (1 + 2)-dimensional space. This has been applied to the classification of all systems of two second-order ODEs admitting four-dimensional symmetry Lie algebras. Moreover, we have shown via examples how this classification can be used for integrating systems of two second-order ODEs admitting four-dimensional symmetry Lie algebras. Furthermore, we have presented examples from applications of systems of two second-order ODEs that admit point symmetry algebra as well as, in relativity, how realizations of vector fields in three variables are utilized in the classification of group of motions. Note that the classification of systems with five symmetries can be achieved by combining our results with the following theorem.

Theorem 2 (Egorov's theorem, see [16]). *Every five-dimensional real Lie algebra contains a four-dimensional real subalgebra.*

A question that needs attention is the study of symmetry breaking for systems of two second-order ODEs: it is well known (see [5]) that a scalar second-order ODE can admit one of 0, 1, 2, 3 or 8 point symmetries. A system of two second-order linear ODEs admits 5, 6, 7, 8 or 15 point symmetries [18]. We conjecture that a system of two second-order ODEs can admit 0, 1, 2, 3, 4, 5, 6, 7, 8 or 15 point symmetries. We will closely pursue this conjecture in another paper.

We have presented physical applications of our results.

6. Summary of results in tables

These are given in tables 1–3. A few important remarks are now in order.

(i) For the functions occurring in table 3, the omitted arguments are just the previous ones.
(ii) The canonical forms in table 3 are not necessarily the simplest ones. There may be changes of variables casting them in simpler forms.

(iii) The realization $\mathcal{L}^7_{4,2}$ gives rise to Riccati equations solvable by quadratures.

(iv) Some systems appearing in table 3 are uncoupled. This fact gives insight into uncoupling systems of two second-order ODEs possessing four-dimensional Lie algebras.

Table 1. Realizations of three-dimensional real Lie algebras in (1 + 2)-space. $l = \partial/\partial t$, $p = \partial/\partial x$, $q = \partial/\partial y$; f, g, h are arbitrary functions.

Algebras	Non-zero brackets	Types	Realizations
		$\mathcal{L}^1_{3,1}$	$X_1 = l,\ X_2 = p,\ X_3 = q$
$\mathcal{L}_{3,1}$		$\mathcal{L}^2_{3,1}$	$X_1 = p,\ X_2 = q,\ X_3 = f(t)p + g(t)q$
		$\mathcal{L}^3_{3,1}$	$X_1 = p,\ X_2 = tp,\ X_3 = yp$
		$\mathcal{L}^1_{3,2}$	$X_1 = -yq,\ X_2 = q,\ X_3 = p$
$\mathcal{L}_{3,2}$	$[X_1, X_2] = X_2$	$\mathcal{L}^2_{3,2}$	$X_1 = -xp - yq,\ X_2 = q,\ X_3 = l$
		$\mathcal{L}^3_{3,2}$	$X_1 = -xp - yq,\ X_2 = q,\ X_3 = f(t)xp + g(t)xq$
		$\mathcal{L}^1_{3,3}$	$X_1 = l,\ X_2 = p,\ X_3 = xl$
$\mathcal{L}_{3,3}$	$[X_2, X_3] = X_1$	$\mathcal{L}^2_{3,3}$	$X_1 = l,\ X_2 = p,\ X_3 = xl + yp$
		$\mathcal{L}^3_{3,3}$	$X_1 = l,\ X_2 = p,\ xl + q$
		$\mathcal{L}^4_{3,3}$	$X_1 = p,\ X_2 = tp,\ X_3 = -l$
		$\mathcal{L}^1_{3,4}$	$X_1 = l,\ X_2 = p,\ X_3 = (t + x)l + xp$
$\mathcal{L}_{3,4}$	$[X_1, X_3] = X_1$	$\mathcal{L}^2_{3,4}$	$X_1 = l,\ X_2 = p,\ X_3 = (t + x)l + xp + q$
	$[X_2, X_3] = X_1 + X_2$	$\mathcal{L}^3_{3,4}$	$X_1 = p,\ X_2 = tp,\ X_3 = -l + xp$
		$\mathcal{L}^1_{3,5}$	$X_1 = l,\ X_2 = p,\ X_3 = tl + xp$
$\mathcal{L}_{3,5}$	$[X_1, X_3] = X_1$	$\mathcal{L}^2_{3,5}$	$X_1 = l,\ X_2 = p,\ X_3 = tl + xp + q$
	$[X_2, X_3] = X_2$	$\mathcal{L}^3_{3,5}$	$X_1 = p,\ X_2 = tp,\ X_3 = xp$
		$\mathcal{L}^4_{3,5}$	$X_1 = p,\ X_2 = tp,\ X_3 = xp + q$
		$\mathcal{L}^{a,1}_{3,6}$	$X_1 = l,\ X_2 = p,\ X_3 = tl + axp$
$\mathcal{L}^a_{3,6}, a \in [-1, 1)$	$[X_1, X_3] = X_1$	$\mathcal{L}^{a,2}_{3,6}$	$X_1 = l,\ X_2 = p,\ X_3 = tl + axp + q$
	$[X_2, X_3] = aX_2$	$\mathcal{L}^{a,3}_{3,6}$	$X_1 = p,\ X_2 = tp,\ X_3 = (1 - a)tl + xp$
		$\mathcal{L}^{a,1}_{3,7}$	$X_1 = l,\ X_2 = p,$
			$X_3 = (at + x)l - (t - ax)p$
$\mathcal{L}^a_{3,7}, a \geqslant 0$	$[X_1, X_3] = aX_1 - X_2$	$\mathcal{L}^{a,2}_{3,7}$	$X_1 = l,\ X_2 = p$
	$[X_2, X_3] = X_1 + aX_2$		$X_3 = (at + x)l - (t - ax)p + q$
		$\mathcal{L}^{a,3}_{3,7}$	$X_1 = p,\ X_2 = tp$
			$X_3 = -(t^2 + 1)l + (a - t)xp$
	$[X_1, X_2] = X_1$	$\mathcal{L}^1_{3,8}$	$X_1 = l,\ X_2 = tl,\ X_3 = -t^2 l$
$\mathcal{L}_{3,8}$	$[X_2, X_3] = X_3$	$\mathcal{L}^2_{3,8}$	$X_1 = l,\ X_2 = tl + xp,$
	$[X_3, X_1] = 2X_1$		$X_3 = -t^2 l - 2xtp + xq$
		$\mathcal{L}^3_{3,8}$	$X_1 = l + p, X_2 = tl + xp,$
			$X_3 = -t^2 l - x^2 p$
		$\mathcal{L}^4_{3,8}$	$X_1 = -tp, X_2 = \frac{1}{2}(-tl + xp),$
			$X_3 = -xl$
	$[X_1, X_2] = X_3$		
$\mathcal{L}_{3,9}$	$[X_3, X_1] = X_2$	$\mathcal{L}^1_{3,9}$	$X_1 = (1 + t^2)l + xtp,\ X_2 = xl - tp$
	$[X_2, X_3] = X_1$		$X_3 = -xtl - (1 + x^2)p$

Table 2. Realizations of four-dimensional real Lie algebras. $l = \partial/\partial t, p = \partial/\partial x, q = \partial/\partial y$; $a, \ldots, d$, are contants, f, g and h are arbitrary functions.

Algebras	Non-zero brackets	Types	Realizations
		$\mathcal{L}^1_{4,1}$	$X_1 = p, X_2 = q, X_3 = f(t)p + g(t)q,$
			$X_4 = h(t)p + k(t)q$
$\mathcal{L}_{4,1}$		$\mathcal{L}^2_{4,1}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = f(t, y)p$
		$\mathcal{L}^1_{4,2}$	$X_1 = -tl - xp - yq, X_2 = p, X_3 = tp, X_4 = yp$
		$\mathcal{L}^2_{4,2}$	$X_1 = -tl, X_2 = l, X_3 = p, X_4 = q$
		$\mathcal{L}^3_{4,2}$	$X_1 = tl, X_2 = tp, X_3 = tl + xp, X_4 = q$
$\mathcal{L}_{4,2}$	$[X_1, X_2] = X_2$	$\mathcal{L}^4_{4,2}$	$X_1 = tl, X_2 = tp, X_3 = p, X_4 = q$
		$\mathcal{L}^5_{4,2}$	$X_1 = -tl - xp, X_2 = p, X_3 = q, X_4 = tp$
		$\mathcal{L}^6_{4,2}$	$X_1 = -xp + f(t)q, X_2 = p, X_3 = q, X_4 = tq$
		$\mathcal{L}^7_{4,2}$	$X_1 = p, X_2 = f(t)\,\mathrm{e}^x p + g(t)\,\mathrm{e}^x q,\ X_3 = q, X_4 = tq$
		$\mathcal{L}^1_{4,3}$	$X_1 = -yq, X_2 = q, X_3 = -tl - xp, X_4 = p$
		$\mathcal{L}^2_{4,3}$	$X_1 = -xp - yq, X_2 = q, X_3 = xp, X_4 = xq$
$\mathcal{L}_{4,3}$	$[X_1, X_2] = X_2$	$\mathcal{L}^3_{4,3}$	$X_1 = -xp - yq, X_2 = q, X_3 = -tl, X_4 = -l$
	$[X_3, X_4] = X_4$	$\mathcal{L}^4_{4,3}$	$X_1 = -tl - xp - yq,\ X_2 = q, X_3 = tl,\ X_4 = tq$
		$\mathcal{L}^5_{4,3}$	$X_1 = xp, X_2 = xq, X_3 = -tl, X_4 = l$
		$\mathcal{L}^6_{4,3}$	$X_1 = tl + xp - yq, X_2 = q, X_3 = tl, X_4 = tp$
		$\mathcal{L}^7_{4,3}$	$X_1 = -p - yq, X_2 = q, X_3 = tl, X_4 = tp$
		$\mathcal{L}^8_{4,3}$	$X_1 = -yq, X_2 = q, X_3 = -xp, X_4 = p$
		$\mathcal{L}^1_{4,4}$	$X_1 = p, X_2 = q, X_3 = yp, X_4 = tp$
		$\mathcal{L}^2_{4,4}$	$X_1 = l, X_2 = p, X_3 = xl, X_4 = q$
$\mathcal{L}_{4,4}$	$[X_2, X_3] = X_1$	$\mathcal{L}^3_{4,4}$	$X_1 = p, X_2 = q, X_3 = l + yp, X_4 = tp$
		$\mathcal{L}^4_{4,4}$	$X_1 = p, X_2 = q, X_3 = yp + f(t)q, X_4 = tp$
		$\mathcal{L}^1_{4,5}$	$X_1 = p, X_2 = tp, X_3 = -l + xp + yq, X_4 = yp$
$\mathcal{L}_{4,5}$	$[X_1, X_3] = X_1$	$\mathcal{L}^2_{4,5}$	$X_1 = p, X_2 = q$
			$X_3 = tl + (x + y)p + yq, X_4 = tp$
	$[X_2, X_3] = X_1 + X_2$	$\mathcal{L}^3_{4,5}$	$X_1 = p,\ X_2 = tp + \mathrm{e}^t q, X_3 = -l + xp,\ X_4 = q$
		$\mathcal{L}^4_{4,5}$	$X_1 = l, X_2 = p, X_3 = (t + x)l + xp, X_4 = q$
	$[X_1, X_3] = X_1$	$\mathcal{L}^{a,1}_{4,6}$	$X_1 = p,\ X_2 = tp,$
			$X_3 = (1 - a)tl + xp + yq,\ X_4 = yp$
$\mathcal{L}^a_{4,6}$	$[X_2, X_3] = aX_2$	$\mathcal{L}^{a,2}_{4,6}$	$X_1 = p, X_2 = q,$
			$X_3 = tl + xp + ayq, X_4 = tp$
$0 < \|a\| \leqslant 1$		$\mathcal{L}^{a,3}_{4,6}$	$X_1 = p, X_2 = q,$
			$X_3 = atl + xp + ayq,\ X_4 = tq$
		$\mathcal{L}^{a,4}_{4,6}$	$X_1 = l, X_2 = p, X_3 = tl + axp,\ X_4 = q$
		$\mathcal{L}^{a,5}_{4,6}$	$X_1 = p - \frac{q}{t},\ X_2 = \frac{q}{t^a}, X_3 = tl + xp, X_4 = tp$
	$[X_1, X_3] = -X_2$	$\mathcal{L}^1_{4,7}$	$X_1 = p, X_2 = tp, X_3 = -(t + 1)l - xp - yq,$
			$X_4 = yp$
$\mathcal{L}_{4,7}$	$[X_2, X_3] = X_1$	$\mathcal{L}^2_{4,7}$	$X_1 = l, X_2 = p, X_3 = xl - tp, X_4 = q$
		$\mathcal{L}^3_{4,7}$	$X_1 = p, X_2 = q, X_3 = l + yp - xq$
			$X_4 = \cos t,\ p - \sin t,\ q$

Table 2. Continued.

Algebras	Non-zero brackets	Types	Realizations
	$[X_1, X_3] = aX_1 - X_2$	$\mathcal{L}_{4,8}^{a,1}$	$X_1 = l,\ X_2 = p,$ $X_3 = (at + x)l - (ax - t)p,\ X_4 = q$
$\mathcal{L}_{4,8}^{a}$ $a > 0$	$[X_2, X_3] = X_1 + aX_2$	$\mathcal{L}_{4,8}^{a,2}$	$X_1 = p, X_2 = q,\ X_3 = l + (ax + y)p + (ay - x)q$ $X_4 = p\cos t - q(\sin t + a\cos t)$
		$\mathcal{L}_{4,9}^{1}$	$X_1 = l, X_2 = tl + xp,$ $X_3 = -t^2 l - 2xtp + xq, X_4 = q$
	$[X_1, X_2] = X_1$	$\mathcal{L}_{4,9}^{2}$	$X_1 = l + p, X_2 = tl + xp,$ $X_3 = -t^2 l - x^2 p, X_4 = q$
$\mathcal{L}_{4,9}$	$[X_2, X_3] = X_3$	$\mathcal{L}_{4,9}^{3}$	$X_1 = -tp, X_2 = \frac{1}{2}(-tl + xp),$ $X_3 = -xl, X_4 = q$
	$[X_3, X_1] = 2X_2$	$\mathcal{L}_{4,9}^{4}$	$X_1 = l, X_2 = tl, X_3 = -t^2 l, X_4 = p$
$\mathcal{L}_{4,10}$	$[X_1, X_2] = X_3$ $[X_3, X_1] = X_2$ $[X_2, X_3] = X_1$	$\mathcal{L}_{4,10}^{1}$	$X_1 = (1 + t^2)l + txp, X_2 = xl - tp,$ $X_3 = -txl - (1 + x^2)p, X_4 = q$
		$\mathcal{L}_{4,11}^{1}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = -l - tq$
$\mathcal{L}_{4,11}$	$[X_2, X_4] = X_1$	$\mathcal{L}_{4,11}^{2}$	$X_1 = p, X_2 = q, X_3 = \frac{-t^2}{2} p - tq, X_4 = l + yp$
	$[X_3, X_4] = X_2$	$\mathcal{L}_{4,11}^{3}$	$X_1 = q, X_2 = p, X_3 = l, X_4 = tp + xq$
	$[X_1, X_4] = aX_1$	$\mathcal{L}_{4,12}^{a,1}$	$X_1 = p, X_2 = tp,$ $X_3 = yp, X_4 = (a - 1)tl + axp + ((a - 1)y - t)q$
$\mathcal{L}_{4,12}^{a}$	$[X_2, X_4] = X_2$	$\mathcal{L}_{4,12}^{a,2}$	$X_1 = p,\ X_2 = q, X_3 = -tq, X_4 = l + axp + yq$
$a \neq 0$	$[X_3, X_4] = X_2 + X_3$	$\mathcal{L}_{4,12}^{a,3}$	$X_1 = l,\ X_2 = p, X_3 = q, X_4 = atl + (x + y)p + yq$
		$\mathcal{L}_{4,12}^{a,4}$	$X_1 = p, X_2 = q, X_3 = \mathrm{e}^{(a-1)t} p - tq,$ $X_4 = l + axp + yq$
		$\mathcal{L}_{4,13}^{1}$	$X_1 = p, X_2 = tp, X_3 = yp,$ $X_4 = tl + xp + (y - t)q$
$\mathcal{L}_{4,13}$	$[X_1, X_4] = X_1$	$\mathcal{L}_{4,13}^{2}$	$X_1 = l, X_2 = p,\ X_3 = q, X_4 = tl + yp$
	$[X_3, X_4] = X_2$	$\mathcal{L}_{4,13}^{3}$	$X_1 = p, X_2 = q, X_3 = a\mathrm{e}^{t} p - tq, X_4 = l + xp$
	$[X_1, X_4] = X_1$	$\mathcal{L}_{4,14}^{1}$	$X_1 = p, X_2 = tp, X_3 = yp,$ $X_4 = -l + xp - tq$
$\mathcal{L}_{4,14}$	$[X_2, X_4] = X_1 + X_2$	$\mathcal{L}_{4,14}^{2}$	$X_1 = q, X_2 = p, X_3 = l,$ $X_4 = tl + (t + x)p + (x + y)q$
	$[X_3, X_4] = X_2 + X_3$	$\mathcal{L}_{4,14}^{3}$	$X_1 = p, X_2 = q,$ $X_3 = -\frac{t^2}{2} p - tq, X_4 = l + (x + y)p + yq$
	$[X_1, X_4] = X_1$	$\mathcal{L}_{4,15}^{a,b,1}$	$X_1 = p,\ X_2 = tp,$ $X_3 = yp,\ X_4 = (1 - a)tl + xp + (1 - b)yq$
$\mathcal{L}_{4,15}^{a,b}$ $-1 \leqslant a < b < 1$	$[X_2, X_4] = aX_2$	$\mathcal{L}_{4,15}^{a,b,2}$	$X_1 = l,\ X_2 = p$ $X_3 = q,\ X_4 = tl + axp + byq$
$ab \neq 0$	$[X_3, X_4] = bX_3$	$\mathcal{L}_{4,15}^{a,b,3}$	$X_1 = p, X_2 = q,$ $X_3 = tp, X_4 = (1 - b)tl + xp + ayq$
		$\mathcal{L}_{4,15}^{a,b,4}$	$X_1 = p, X_2 = q,$ $X_3 = tq, X_4 = (a - b)tl + xp + ayq$
		$\mathcal{L}_{4,15}^{a,b,5}$	$X_1 = p, X_2 = q,$ $X_3 = \mathrm{e}^{(1-b)t} p + \mathrm{e}^{(a-b)t} q, X_4 = l + xp + ayq$

Table 2. Continued.

Algebras	Non-zero brackets	Types	Realizations
	$[X_1, X_4] = X_1$	$\mathcal{L}^{a,a,1}_{4,15}$	$X_1 = p,\ X_2 = tp,$
			$X_3 = yp,\ X_4 = (1-a)tl + xp + (1-a)yq$
$\mathcal{L}^{a,a}_{4,15}$	$[X_2, X_4] = aX_2$	$\mathcal{L}^{a,a,2}_{4,15}$	$X_1 = l,\ X_2 = p$
$-1 \leqslant a < 1$			$X_3 = q,\ X_4 = tl + axp + ayq$
$a \neq 0$	$[X_3, X_4] = aX_3$	$\mathcal{L}^{a,a,3}_{4,15}$	$X_1 = p, X_2 = q,$
			$X_3 = tp, X_4 = (1-a)tl + xp + ayq$
		$\mathcal{L}^{a,a,4}_{4,15}$	$X_1 = p, X_2 = q, X_3 = tq, X_4 = xp + ayq$
	$[X_1, X_4] = X_1$	$\mathcal{L}^{a,1,1}_{4,15}$	$X_1 = p,\ X_2 = tp,\ X_3 = yp,$
			$X_4 = (1-a)tl + xp$
$\mathcal{L}^{a,1}_{4,15}$	$[X_2, X_4] = aX_2$	$\mathcal{L}^{a,1,2}_{4,15}$	$X_1 = l,\ X_2 = p$
$-1 \leqslant a < 1$			$X_3 = q,\ X_4 = tl + axp + yq$
$a \neq 0$	$[X_3, X_4] = X_3$	$\mathcal{L}^{a,1,3}_{4,15}$	$X_1 = p, X_2 = qX_3 = tp, X_4 = xp + ayq$
		$\mathcal{L}^{a,1,4}_{4,15}$	$X_1 = p, X_2 = q, X_3 = tq,$
			$X_4 = (a-1)tl + xp + ayq$
	$[X_1, X_4] = X_1$	$\mathcal{L}^{1}_{4,16}$	$X_1 = l, X_2 = p, X_3 = q, X_4 = tl + xp + yq$
$\mathcal{L}_{4,16}$	$[X_2, X_4] = X_2$	$\mathcal{L}^{2}_{4,16}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = xp$
	$[X_3, X_4] = X_3$	$\mathcal{L}^{3}_{4,16}$	$X_1 = p, X_2 = q, X_3 = f(t)p + g(t)q,$
			$X_4 = xp + yq$
	$[X_1, X_4] = aX_1$	$\mathcal{L}^{a,b,1}_{4,17}$	$X_1 = p, X_2 = tp,\ X_3 = yp,$
			$X_4 = ((a-b)t + y)l + axp + ((a-b)y - t)q$
$\mathcal{L}^{a,b}_{4,17}$	$[X_2, X_4] = bX_2 - X_3$	$\mathcal{L}^{a,b,2}_{4,17}$	$X_1 = q, X_2 = p, X_3 = l$
$a \neq 0, b \geqslant 0$			$X_4 = (bt - x)l + (bx + t)p + ayq$
	$[X_3, X_4] = X_2 + bX_3$	$\mathcal{L}^{a,b,3}_{4,17}$	$X_1 = p, X_2 = q,\ X_3 = c\dfrac{e^{(a-b)t}}{\cos t}p - \tan tq,$
			$X_4 = l + \left(ax - cy\dfrac{e^{(a-b)t}}{\cos t}\right)p + (b + \tan t)yq$
	$[X_1, X_4] = 2X_1$	$\mathcal{L}^{1}_{4,18}$	$X_1 = 2p, X_2 = 2tp, X_3 = -l,$
			$X_1 = tl + (2x - t^2)p$
	$[X_2, X_4] = X_2$	$\mathcal{L}^{2}_{4,18}$	$X_1 = 2p, X_2 = 2tp, X_3 = -l,$
			$X_4 = tl + (2x - t^2)p + q$
$\mathcal{L}_{4,18}$	$[X_3, X_4] = X_2 + X_3$	$\mathcal{L}^{3}_{4,18}$	$X_1 = 2q,\ X_2 = p, X_3 = l + 2xq,$
			$X_4 = tl + (t + x)p + (2y + t^2)q$
	$[X_2, X_3] = X_1$	$\mathcal{L}^{4}_{4,18}$	$X_1 = q, X_2 = p, X_3 = -tp + xq,$
			$X_4 = l + xp + 2yq$
		$\mathcal{L}^{1}_{4,19}$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = xp + tq$
		$\mathcal{L}^{2}_{4,19}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = -tl$
	$[X_2, X_3] = X_1$	$\mathcal{L}^{3}_{4,19}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = -tl + yp$
$\mathcal{L}_{4,19}$	$[X_2, X_4] = X_2$	$\mathcal{L}^{4}_{4,19}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = -tl + q$
	$[X_3, X_4] = -X_3$	$\mathcal{L}^{5}_{4,19}$	$X_1 = q, X_2 = p, X_3 = xq,\ X_4 = 2tl + xp$
	$[X_3, X_4] = -X_3$	$\mathcal{L}^{6}_{4,19}$	$X_1 = q, X_2 = p, X_3 = tp + xq,\ X_4 = 2tl + xp$
		$\mathcal{L}^{7}_{4,19}$	$X_1 = q, X_2 = p, X_3 = l + xq, X_4 = -tl + xp$
		$\mathcal{L}^{8}_{4,19}$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = xp$

Table 2. Continued.

Algebras	Non-zero brackets	Types	Realizations
		$\mathcal{L}^{b,1}_{4,20}$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = xp + (1+b)yq$
	$[X_2, X_3] = X_1$	$\mathcal{L}^{b,2}_{4,20}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = btl + (1+b)xp$
$\mathcal{L}^{b}_{4,20}$	$[X_1, X_4] = (1+b)X_1$	$\mathcal{L}^{b,3}_{4,20}$	$X_1 = p, X_2 = tp, X_3 = -l,$
			$X_4 = btl + (1+b)xp + q$
$b \in [0, 1]$	$[X_2, X_4] = X_2$	$\mathcal{L}^{b,4}_{4,20}$	$X_1 = q, X_2 = p, X_3 = xq,$
			$X_4 = (1-b)tl + xp + (1+b)yq$
		$\mathcal{L}^{b,5}_{4,20}$	$X_1 = q, X_2 = p, X_3 = tp + xq,$
			$X_4 = (1-b)tl + xp + (1+b)yq$
	$[X_3, X_4] = bX_3$	$\mathcal{L}^{b,6}_{4,20}$	$X_1 = q, X_2 = p, X_3 = l + xq,$
			$X_4 = btl + xp + (1+b)yq$
	$[X_2, X_3] = X_1$	$\mathcal{L}^{a,1}_{4,21}$	$X_1 = q, X_2 = p, X_3 = l + xq,$
	$[X_1, X_4] = 2aX_1$		$X_4 = (at - x + b)l + (ax + t)p$
			$+ \left(2ay + \frac{1}{2}(t^2 - x^2)\right) q$
$\mathcal{L}^{a}_{4,21}$	$[X_2, X_4] = aX_2 - X_3$	$\mathcal{L}^{a,2}_{4,21}$	$X_1 = q, X_2 = p, X_3 = -tp + xq,$
$a \geqslant 0$	$[X_3, X_4] = X_2 + aX_3$		$X_4 = (1+t^2)l + (a+t)xp + \left(2ay - \dfrac{x^2}{2}\right) q$
		$\mathcal{L}^{1}_{4,22}$	$X_1 = p, X_2 = tp, X_3 = xp, X_4 = -(1+t^2)l - txp$
$\mathcal{L}_{4,22}$	$[X_1, X_3] = X_1$	$\mathcal{L}^{2}_{4,22}$	$X_1 = l, X_2 = p, X_3 = tl + xp, X_4 = xl - tp$
	$[X_1, X_4] = -X_2$	$\mathcal{L}^{3}_{4,22}$	$X_1 = p, X_2 = tp, X_3 = xp + yq,$
			$X_4 = -(1+t^2)l - txp$
	$[X_2, X_4] = X_1$	$\mathcal{L}^{4}_{4,22}$	$X_1 = l, X_2 = p, X_3 = tl + xp, \ X_4 = xl - tp + q$
		$\mathcal{L}^{5}_{4,22}$	$X_1 = q, X_2 = p, X_3 = tl + xp + yq,$
			$X_4 = ctl - yp + xq$

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Table 3. Classification of systems of two second-order odes admitting four-dimensional Lie algebras. $l = \partial/\partial t$, $p = \partial/\partial x$, $q = \partial/\partial y$; , f, g, η, μ are arbitrary functions.

Algebras	Types	Realizations	Equations
	$\mathcal{L}^1_{4,2}$	$X_1 = -tl - xp - yq,\ X_2 = p,\ X_3 = tp,\ X_4 = yp$	$\ddot{x} = t^{-1} f(y/t, \dot{y}),\ \ddot{y} = 0$
	$\mathcal{L}^2_{4,2}$	$X_1 = -tl,\ X_2 = l,\ X_3 = p,\ X_4 = q$	$\ddot{x} = \dot{x}^2 f(\dot{x}/\dot{y}),\ \ddot{y} = \dot{x}^2 g(\dot{x}/\dot{y})$
	$\mathcal{L}^3_{4,2}$	$X_1 = tl,\ X_2 = tp,\ X_3 = tl + xp,\ X_4 = q$	$\ddot{x} = \dot{x}\dot{y} f(t\dot{y}),\ \ddot{y} = \dot{y}^2 g(t\dot{y})$
	$\mathcal{L}^4_{4,2}$	$X_1 = tl,\ X_2 = tp,\ X_3 = p,\ X_4 = q$	$t^2\ddot{x} = f(t\dot{y}),\ t^2\ddot{y} = g(t\dot{y})$
	$\mathcal{L}^5_{4,2}$	$X_1 = -tl - xp,\ X_2 = p,\ X_3 = q,\ X_4 = tp$	$t\ddot{x} = f(t\dot{y}),\ t^2\ddot{y} = g(t\dot{y})$
$\mathcal{L}_{4,2}$	$\mathcal{L}^6_{4,2}$	$X_1 = -xp + \mu(t)q,\ X_2 = p,\ X_3 = q,\ X_4 = tq,\ \ddot{\mu} \neq 0$	$\ddot{x} = f(t)\dot{x},\ \ddot{y} = -\ddot{\mu}(t)\ln\dot{x} + g(t)$
	$\mathcal{L}^7_{4,2}$	$X_1 = p,\ X_2 = \mu(t)\,\mathrm{e}^x q,\ X_3 = q,\ X_4 = tq,\ \mu \neq 0$	$\ddot{x} = -\dot{x}^2 - \frac{2\dot{\mu}}{\mu}\dot{x} - \frac{\ddot{\mu}}{\mu},\ \ddot{y} = g(t, \dot{x})$
		$X_1 = p,\ X_2 = \eta(t)\,\mathrm{e}^x p + \mu(t)\,\mathrm{e}^x q,\ X_3 = q,\ X_4 = tq,\ \eta \neq 0$	$\ddot{x} = \dot{x}^2 + \left(\frac{\dot{\eta}}{\eta} + \eta f(t)\right)\dot{x} + f(t)\dot{\eta} - \frac{\ddot{\eta}}{\eta} + \frac{\dot{\eta}^2}{\eta^2}$ $\ddot{y} = \frac{\mu}{\eta}\dot{x}^2 + \left(2\frac{\dot{\mu}}{\eta} - \frac{\mu\dot{\eta}}{\eta^2} + \mu f(t)\right)\dot{x}$ $+\left(\frac{\ddot{\mu}}{\eta} - \frac{\mu\ddot{\eta}}{\eta^2} - 2\frac{\dot{\eta}\dot{\mu}}{\eta^2}\right)\ln(\eta\dot{x} + \dot{\eta}) + g(t)$
	$\mathcal{L}^1_{4,3}$	$X_1 = -yq,\ X_2 = q,\ X_3 = -tl - xp,\ X_4 = p$	$t\ddot{x} = f(\dot{x}),\ \ddot{y} = \dot{y}g(\dot{x})$
	$\mathcal{L}^2_{4,3}$	$X_1 = -xp - yq,\ X_2 = q,\ X_3 = xp,\ X_4 = xq$	$\ddot{x} = xf(t, \dot{x}/x),\ \dot{x}\ddot{y} = x\dot{y} f(t, \dot{x}/x)$
	$\mathcal{L}^3_{4,3}$	$X_1 = -xp - yq,\ X_2 = q,\ X_3 = -tl,\ X_4 = -l$	$x\ddot{x} = \dot{x}^2 f(\dot{x}/\dot{y}),\ x\ddot{y} = \dot{x}^2 g(\dot{x}/\dot{y})$
$\mathcal{L}_{4,3}$	$\mathcal{L}^4_{4,3}$	$X_1 = -tl - xp - yq,\ X_2 = q,\ X_3 = tl,\ X_4 = tq$	$t^2 x\ddot{x} = f(t\dot{x}/x),\ t^2 x\ddot{y} = g(t\dot{x}/x)$
	$\mathcal{L}^5_{4,3}$	$X_1 = xp,\ X_2 = xq,\ X_3 = -tl,\ X_4 = l$	$x\ddot{x} = \dot{x}^2 f(y - x\dot{y}/\dot{x}),\ x^2\ddot{y} = (yf(y - x\dot{y}/\dot{x}) + g(y - x\dot{y}/\dot{x}))$
	$\mathcal{L}^6_{4,3}$	$X_1 = tl + xp - yq,\ X_2 = q,\ X_3 = tl,\ X_4 = tp$	$t^2\ddot{x} = (t\dot{x} - x) f\,(t\dot{y}(t\dot{x} - x)),\ t\ddot{y} = \dot{y}g\,(t\dot{y}(t\dot{x} - x))$
	$\mathcal{L}^7_{4,3}$	$X_1 = -p - yq,\ X_2 = q,\ X_3 = tl,\ X_4 = tp$	$t^2\ddot{x} = f\left(t\dot{y}\mathrm{e}^{(t\dot{x}-x)}\right),\ t\ddot{y} = \dot{y}g\left(t\dot{y}\mathrm{e}^{(t\dot{x}-x)}\right)$

Table 3. Continued.

Algebras	Types	Realizations	Equations
	$\mathcal{L}^1_{4,4}$	$X_1 = p, X_2 = q, X_3 = yp, X_4 = tp$	$\ddot{x} = f(t, \dot{y}), \ddot{y} = 0$
	$\mathcal{L}^2_{4,4}$	$X_1 = l, X_2 = p, X_3 = xl, X_4 = q$	$\ddot{x} = \dot{x}^3 f(\dot{x}/\dot{y}), \ddot{y} = \dot{x}^2\dot{y} f(\dot{x}/\dot{y}) + \dot{y}^2 g(\dot{x}/\dot{y})$
$\mathcal{L}_{4,4}$	$\mathcal{L}^3_{4,4}$	$X_1 = p, X_2 = q, X_3 = l + yp, X_4 = tp$	$\ddot{x} = f(\dot{y}) + tg(\dot{y}), \ddot{y} = g(\dot{y})$
	$\mathcal{L}^4_{4,4}$	$X_1 = p, X_2 = q, X_3 = yp + \mu(t)q, X_4 = tp,\ \ \ddot{\mu} \neq 0$	$\ddot{x} = \dfrac{\dddot{\mu}}{2\ddot{\mu}^2}\dot{y}^2 + g(t)\dot{y} + f(t), \ddot{y} = \dfrac{\dddot{\mu}}{\ddot{\mu}}\dot{y} + g(t)$
	$\mathcal{L}^1_{4,5}$	$X_1 = p, X_2 = tp, X_3 = -l + xp + yq, X_4 = yp$	$\ddot{x} = \mathrm{e}^{-t} f(y\mathrm{e}^t, \dot{y}/y), \ddot{y} = 0$
$\mathcal{L}_{4,5}$	$\mathcal{L}^2_{4,5}$	$X_1 = p, X_2 = q, X_3 = tl + (x + y)p + yq, X_4 = tp$	$2t\ddot{x} = 2t^2 f(\dot{y}) - g(\dot{y}), t\ddot{y} = g(\dot{y})$
	$\mathcal{L}^3_{4,5}$	$X_1 = p, X_2 = tp + \mathrm{e}^t q, X_3 = -l + xp, X_4 = q$	$\ddot{x} = \mathrm{e}^{-t} f(\dot{y} + \dot{x}\mathrm{e}^t), \ddot{y} = g(\dot{y} + \dot{x}\mathrm{e}^t)$
	$\mathcal{L}^4_{4,5}$	$X_1 = l, X_2 = p, X_3 = (t + x)l + xp, X_4 = q$	$\ddot{x} = \dot{x}^3\mathrm{e}^{-1/\dot{x}} f(\dot{y}/\dot{x}\mathrm{e}^{1/\dot{x}}),$ $\ddot{y} = \dot{x}^2\dot{y}\mathrm{e}^{-1/\dot{x}} f(\dot{y}/\dot{x}\mathrm{e}^{1/\dot{x}}) + \dot{x}^2\mathrm{e}^{-2/\dot{x}} g(\dot{y}/\dot{x}\mathrm{e}^{1/\dot{x}})$
	$\mathcal{L}^{a,1}_{4,6}$	$X_1 = p, X_2 = tp, X_3 = (a - 1)tl + xp + yq, X_4 = yp$	$\ddot{x} = y^{2a-1} f(ty^{a-1}, \dot{y}/y^a), \ddot{y} = 0$
	$\mathcal{L}^{a,2}_{4,6}$	$X_1 = p, X_2 = q, X_3 = tl + xp + ayq, X_4 = tp$	$t\ddot{x} = f(t^{a-1}\dot{y}), \ddot{y} = t^{a-2} g(t^{a-1}\dot{y})$
$\mathcal{L}^a_{4,6}$	$\mathcal{L}^{a,3}_{4,6}$	$X_1 = p, X_2 = q, X_3 = atl + xp + ayq, X_4 = tq$	$\ddot{x} = t^{\frac{1-2a}{a}} f(t^{\frac{a-1}{a}}\dot{x}), t\ddot{y} = g(t^{\frac{a-1}{a}}\dot{x})$
$0 < \lvert a\rvert \leqslant 1$	$\mathcal{L}^{1,4}_{4,6}$	$X_1 = l, X_2 = p, X_3 = tl + xp, X_4 = q$	$\dot{y}\ddot{x} = f(\dot{x}), \ddot{y} = \dot{y}^2 g(\dot{x})$
	$\mathcal{L}^{a,4}_{4,6}, a \neq 1$	$X_1 = l, X_2 = p, X_3 = tl + axp, X_4 = q$	$\ddot{x} = \dot{x}^{\frac{a-2}{a-1}} f(\dot{x}\dot{y}^{a-1}), \ddot{y} = \dot{x}^{\frac{2}{1-a}} g(\dot{x}\dot{y}^{a-1})$
	$\mathcal{L}^{a,5}_{4,6}$	$X_1 = p - t^{-1}q, X_2 = t^{-a}q, X_3 = tl + xp, X_4 = tp$	$t\ddot{x} = f\,(\exp(-x/t + (1 - a)(ay + t\dot{y}))),$ $t^2\ddot{y} = (2 - a)(a - 1)(\ln\dot{x} - x/t)$ $\quad + g\,(\exp(-x/t + (1 - a)(ay + t\dot{y})))$
	$\mathcal{L}^1_{4,7}$	$X_1 = p, X_2 = tp, X_3 = -(1 + t)l - xp - yq, X_4 = yp$	$(1 + t)\ddot{x} = f(y/(1 + t), \dot{y}), \ddot{y} = 0$
$\mathcal{L}_{4,7}$	$\mathcal{L}^2_{4,7}$	$X_1 = l, X_2 = p, X_3 = xl - tp, X_4 = q$	$\ddot{x} = (1 + \dot{x}^2)^{3/2} f(\dot{y}^2/(1 + \ddot{x}^2)),$ $\ddot{y} = (1 + \dot{x}^2)^{1/2}\dot{x}\dot{y} f(\ ) + (1 + \dot{x}^2) g(\ )$

Table 3. Continued.

Algebras	Types	Realizations	Equations
$\mathcal{L}_{4,7}$	$\mathcal{L}_{4,7}^{3}$	$X_1 = p, X_2 = q, X_3 = l + yp - xq, X_4 = p\cos t - q\sin t$	$\ddot{x} = -f(v\cos\varphi)\sin(t + g(v\cos\varphi)) + v\sin\varphi\cos t,$ $\ddot{y} = f(\,)\cos(t + g(\,)) + v\sin\varphi\sin t$ $v = (\dot{x}^2 + \dot{y}^2)^{1/2}, \varphi = t + \arctan\dot{y}/\dot{x}$
$\mathcal{L}_{4,8}^{a}$ $a > 0$	$\mathcal{L}_{4,8}^{a,1}$	$X_1 = l, X_2 = p, X_3 = (at + x)l - (ax - t)p, X_4 = q$	$\ddot{x} = (1 + \dot{x}^2)^{3/2}\mathrm{e}^{a\arctan\dot{x}} f\left(\frac{\dot{y}^2}{1 + \dot{x}^2}\mathrm{e}^{-2a\arctan\dot{x}}\right),$ $\ddot{y} = (1 + \dot{x}^2)^{1/2}\dot{x}\dot{y}\mathrm{e}^{a\arctan\dot{x}} f(\,) + (1 + \dot{x}^2)\,\mathrm{e}^{2a\arctan\dot{x}} g(\,)$
$\mathcal{L}_{4,9}$	$\mathcal{L}_{4,9}^{1}$	$X_1 = l, X_2 = tl + xp, X_3 = -t^2 l - 2xtp + xq, X_4 = q$	$2x\ddot{x} = \dot{x}^2 + f\left(\frac{\dot{x}^2}{2} + 2x\dot{y}\right)$ $2x^2\ddot{y} = -\dot{x}\left(\frac{\dot{x}^2}{2} + 2x\dot{y}\right) - \dot{x} f(\,) + g(\,)$
	$\mathcal{L}_{4,9}^{2}$	$X_1 = l + p, X_2 = tl + xp, X_3 = -t^2 l - x^2 p, X_4 = q$	$\ddot{x} = -\frac{2\dot{x}^2}{t - x} + \frac{2\dot{x}}{3\dot{y}^2(t - x)^3} + \dot{y}^3(t - x)^2 f\left(\frac{\dot{x}}{\dot{y}^2(t - x)^2}\right)$ $\ddot{y} = -\frac{2}{3\dot{y}(t - x)^3} + \dot{y}^2 g(\,)$
	$\mathcal{L}_{4,11}^{1}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = -l - tq$	$\ddot{x} = f(t^2 - 2y, t - \dot{y}), \ddot{y} = 0$
$\mathcal{L}_{4,11}$	$\mathcal{L}_{4,11}^{2}$	$X_1 = p, X_2 = q, X_3 = -\frac{1}{2}p - tq, X_4 = l + yp$	$\ddot{x} = \dot{y} + f(t\dot{y} - \dot{x}) + tg(t\dot{y} - \dot{x}), \ddot{y} = g(t\dot{y} - \dot{x})$
	$\mathcal{L}_{4,11}^{3}$	$X_1 = q, X_2 = p, \ X_3 = l, X_4 = tp + xq$	$\ddot{x} = f(\dot{x}^2 + 2\dot{y}), \ddot{y} = \dot{x} f(\dot{x}^2 + 2\dot{y}) + g(\dot{x}^2 + 2\dot{y})$
	$\mathcal{L}_{4,12}^{a,1}, a \neq 1$	$X_1 = p, X_2 = tp, X_3 = yp,$ $X_4 = (a - 1)tl + axp + ((a - 1)y - t)q$	$\ddot{x} = \mathrm{e}^{(a-2)\dot{y}} f(t\mathrm{e}^{(a-1)\dot{y}}, t^{\frac{1}{a-1}}\mathrm{e}^{y/t}), \ddot{y} = 0$
	$\mathcal{L}_{4,12}^{1,1}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = xp - tq$	$\ddot{x} = \mathrm{e}^{-\dot{y}} f(t, t\dot{y} - y), \ddot{y} = 0$
$\mathcal{L}_{4,12}^{a}, a \neq 0$	$\mathcal{L}_{4,12}^{a,2}$	$X_1 = p, X_2 = q, X_3 = -tq, X_4 = l + axp + yq$	$\ddot{x} = \dot{x} f(\dot{x}\mathrm{e}^{-at}), \ddot{y} = \dot{x}^{1/a} g(\dot{x}\mathrm{e}^{-at})$
	$\mathcal{L}_{4,12}^{a,3}, a \neq 1$	$X_1 = l, X_2 = p, X_3 = q, X_4 = atl + (x + y)p + yq$	$\ddot{x} = \frac{1}{a - 1}\dot{y}^{\frac{1-2a}{1-a}}\ln\dot{y}\, g(\dot{y}^{\frac{1}{a-1}}\mathrm{e}^{\dot{x}/\dot{y}}) + \dot{y}^{\frac{1-2a}{1-a}} f(\,),$ $\ddot{y} = \dot{y}^{\frac{1-2a}{1-a}} g(\,)$

Table 3. Continued.

Algebras	Types	Realizations	Equations
$\mathcal{L}^{a}_{4,12}, a \neq 0$	$\mathcal{L}^{1,3}_{4,12}$	$X_1 = l, X_2 = p, X_3 = q, X_4 = tl + (x + y)p + yq$	$\dot{y}\ddot{x} = \mathrm{e}^{-\dot{x}/\dot{y}}(\dot{y}f(\dot{y}) + \dot{x}g(\dot{y})),\ \ddot{x} = \mathrm{e}^{-\dot{x}/\dot{y}}g(\dot{y})$
	$\mathcal{L}^{1}_{4,13}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = tl + xp + (y - t)q$	$\ddot{x} = \mathrm{e}^{\dot{y}}f(t\mathrm{e}^{y/t}, t\mathrm{e}^{\dot{y}}), \ddot{y} = 0$
$\mathcal{L}_{4,13}$	$\mathcal{L}^{2}_{4,13}$	$X_1 = l, X_2 = p, X_3 = q, X_4 = tl + yp$	$\dot{y}\ddot{x} = f(\dot{y}\mathrm{e}^{\dot{x}/\dot{y}}) - \frac{1}{4}\dot{y}^4 g(\dot{y}\mathrm{e}^{\dot{x}/\dot{y}}), \ddot{y} = \dot{y}^2 g(\dot{y}\mathrm{e}^{\dot{x}/\dot{y}})$
	$\mathcal{L}^{3}_{4,13}$	$X_1 = p, X_2 = q, X_3 = c\mathrm{e}^{t}p - tq,\ X_4 = l + xp$	$\ddot{x} = \mathrm{e}^{t}(f(c\dot{y} + \dot{x}\mathrm{e}^{-t}) - c\dot{y}), \ddot{y} = g(c\dot{y} + \dot{x}\mathrm{e}^{-t})$
	$\mathcal{L}^{1}_{4,14}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = -l + xp - tq$	$\ddot{x} = \mathrm{e}^{-t}f(t^2 - 2y, (\dot{y} - 1)\,\mathrm{e}^{t}), \ddot{y} = 0$
$\mathcal{L}_{4,14}$	$\mathcal{L}^{2}_{4,14}$	$X_1 = q, X_2 = p, X_3 = l, X_4 = tl + (t + x)p + (x + y)q$	$\ddot{x} = \mathrm{e}^{-\dot{x}}f(\dot{x}^2 - 2\dot{y}),$ $\ddot{y} = \dot{x}\mathrm{e}^{-\dot{x}}f(\dot{x}^2 - 2\dot{y}) + \mathrm{e}^{-\dot{x}}g(\dot{x}^2 - 2\dot{y})$
	$\mathcal{L}^{3}_{4,14}$	$X_1 = p, X_2 = q, X_3 = \dfrac{-t^2}{2}p - tq, X_4 = l + (x + y)p + yq$	$\ddot{x} = \dot{y} + \mathrm{e}^{t}\left(f(\mathrm{e}^{-t}(\dot{x} - t\dot{y})) + tg(\mathrm{e}^{-t}(\dot{x} - t\dot{y}))\right),$ $\ddot{y} = \mathrm{e}^{t}g(\mathrm{e}^{-t}(\dot{x} - t\dot{y}))$
	$\mathcal{L}^{a,b,1}_{4,15}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = (1 - a)tl + xp + (1 - b)yq$	$\ddot{x} = t^{\frac{2a-1}{a-1}}f(ty^{\frac{a-1}{1-b}}, t\dot{y}^{\frac{a-1}{a-b}}),\ \ddot{y} = 0$
$\mathcal{L}^{a,b}_{4,15}$	$\mathcal{L}^{a,b,2}_{4,15}$	$X_1 = l, X_2 = p, X_3 = q, X_4 = tl + axp + byq$	$\ddot{x} = \dot{x}^{\frac{2-a}{1-a}}f(\dot{x}\dot{y}^{\frac{a-1}{1-b}}), \ddot{y} = \dot{x}^{\frac{2-b}{1-a}}g(\dot{x}\dot{y}^{\frac{a-1}{1-b}})$
$-1 \leqslant a < b < 1$	$\mathcal{L}^{a,b,3}_{4,15}$	$X_1 = p, X_2 = q, X_3 = tp, X_4 = (1 - b)tl + xp + ayq$	$\ddot{x} = t^{\frac{2b-1}{1-b}}f(\dot{y}t^{\frac{a+b-1}{b-1}}), \ddot{y} = t^{\frac{a+2b-2}{1-b}}g(\dot{y}t^{\frac{a+b-1}{b-1}})$
$ab \neq 0$	$\mathcal{L}^{a,b,4}_{4,15}$	$X_1 = p, X_2 = q, X_3 = tq, X_4 = (a - b)tl + xp + ayq$	$\ddot{x} = t^{\frac{1-2a+2b}{a-b}}f(\dot{x}t^{\frac{1+b-a}{b-a}}), \ddot{y} = t^{\frac{2b-a}{a-b}}g(\dot{x}t^{\frac{1+b-a}{b-a}})$
	$\mathcal{L}^{a,a,1}_{4,15}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = (1 - a)tl + xp + (1 - a)yq$	$\ddot{x} = t^{\frac{2a-1}{a-1}}f(t/y, \dot{y}), \ddot{y} = 0$
$\mathcal{L}^{a,a}_{4,15}$	$\mathcal{L}^{a,a,2}_{4,15}$	$X_1 = l, X_2 = p, X_3 = q, X_4 = tl + axp + ayq$	$\ddot{x} = \dot{x}^{\frac{2-a}{1-a}}f(\dot{x}/\dot{y}), \ddot{y} = \dot{x}^{\frac{2-a}{1-a}}g(\dot{x}/\dot{y})$
$-1 \leqslant a < 1$	$\mathcal{L}^{a,a,3}_{4,15}$	$X_1 = p, X_2 = q, X_3 = tp, X_4 = (1 - a)tl + xp + ayq$	$\ddot{x} = \dot{y}f(\dot{y}t^{\frac{1-2a}{1-a}}),\ \ddot{y} = t^{\frac{3a-2}{1-a}}g(\dot{y}t^{\frac{1-2a}{1-a}})$
$a \neq 0$	$\mathcal{L}^{a,a,4}_{4,15}$	$X_1 = p, X_2 = q, X_3 = tq, X_4 = xp + ayq$	$\ddot{x} = f(t)\dot{x}, \ddot{y} = g(t)\dot{x}^{a}$

Table 3. Continued.

Algebras	Types	Realizations	Equations
	$\mathcal{L}_{4,15}^{a,1,1}$	$X_1 = p, X_2 = tp, X_3 = yp, X_4 = (1-a)tl + xp$	$\ddot{x} = t^{\frac{2a-1}{a-1}} f(y, t\dot{y}), \ddot{y} = 0$
$\mathcal{L}_{4,15}^{a,1}$	$\mathcal{L}_{4,15}^{a,1,2}$	$X_1 = l, X_2 = p, X_3 = q, X_4 = tl + axp + yq$	$\ddot{x} = \dot{x}^{\frac{2-a}{1-a}} f(\dot{y}), \ddot{y} = \dot{x}^{\frac{1}{1-a}} g(\dot{y})$
$-1 \leqslant a < 1$	$\mathcal{L}_{4,15}^{a,1,3}$	$X_1 = p, X_2 = q, X_3 = tp, X_4 = xp + ayq$	$\ddot{x} = f(t)\dot{y}^{1/a}, \ddot{y} = g(t)\dot{y}$
$a \neq 0$	$\mathcal{L}_{4,15}^{a,1,4}$	$X_1 = p, X_2 = q, X_3 = tq, X_4 = (a-1)tl + xp + ayq$	$\ddot{x} = t^{\frac{2a-3}{1-a}} f(\dot{x}t^{\frac{a-2}{a-1}}), \ddot{y} = t^{\frac{a-2}{1-a}} g(\dot{x}t^{\frac{a-2}{a-1}})$
	$\mathcal{L}_{4,17}^{a,b,1}$	$X_1 = p, X_2 = tp, X_3 = yp,$ $X_4 = ((a-b)t + y)l + axp + ((a-b)y - t)q$	$\ddot{x} = (1+\dot{y}^2)\,\mathrm{e}^{(a-2b)\arctan\dot{y}}$ $f((t^2+y^2)^{1/2}\mathrm{e}^{(a-b)\arctan y/t}, (t+y\dot{y})/(y-t\dot{y})), \ddot{y} = 0$
$\mathcal{L}_{4,17}^{a,b}$ $a \neq 0, b \geqslant 0$	$\mathcal{L}_{4,17}^{a,b,2}$	$X_1 = q, X_2 = p, X_3 = l,$ $X_4 = (bt - x)l + (bx + t)p + ayq$	$\ddot{x} = (1+\dot{x})^{3/2}\mathrm{e}^{-b\arctan\dot{x}} f(\dot{y}(1+\dot{x}^2)^{-1/2}\mathrm{e}^{(a-b)\arctan\dot{x}}),$ $\ddot{y} = \dot{x}\dot{y}(1+\dot{x}^2)^{1/2}\mathrm{e}^{(a-b)\arctan\dot{x}} f(\,) + (1+\dot{x}^2)\,\mathrm{e}^{(2a-b)\arctan\dot{x}} g(\,)$
	$\mathcal{L}_{4,17}^{a,b,3}$	$X_1 = p, X_2 = q, X_3 = -tq, X_4 = l + axp + (b+t)yq$	$\ddot{x} = -\dfrac{2t\dot{x}}{1+t^2} + \dfrac{\mathrm{e}^{-2a\arctan t}}{(1+t^2)^2} f(\dot{x}(1+t^2)\,\mathrm{e}^{-a\arctan t}),$ $\ddot{y} = (1+t^2)^{-3/2}\mathrm{e}^{b\arctan t} g(\,)$
	$\mathcal{L}_{4,18}^{1}$	$X_1 = 2p, X_2 = 2tp, X_3 = -l, X_4 = tl + (2x - t^2)p$	$\ddot{x} = 2\ln\dot{y} + f(y), \ddot{y} = g(y)\dot{y}^2$
$\mathcal{L}_{4,18}$	$\mathcal{L}_{4,18}^{2}$	$X_1 = 2p, X_2 = 2tp, X_3 = -l, X_4 = tl + (2x - t^2)p + q$	$\ddot{x} = -2y + f(\mathrm{e}^y\dot{y}), \ddot{y} = \dot{y}^2 g(\mathrm{e}^y\dot{y})$
	$\mathcal{L}_{4,18}^{3}$	$X_1 = 2q, X_2 = p, X_3 = l + 2xq,$ $X_4 = tl + (t+x)p + (2y + t^2)q$	$\ddot{x} = \mathrm{e}^{-\dot{x}} f(\mathrm{e}^{-\dot{x}}(t\dot{x} - \dot{y}/2)), \ddot{y} = 2\dot{x} + \mathrm{e}^{-\dot{x}} g(\mathrm{e}^{-\dot{x}}(t\dot{x} - \dot{y}/2))$
	$\mathcal{L}_{4,18}^{4}$	$X_1 = q, X_2 = p, X_3 = -tp + xq, X_4 = l + xp + 2yq$	$\ddot{x} = \mathrm{e}^t f(\mathrm{e}^{-2t}(\dot{x}^2 + 2\dot{y})),$ $\ddot{y} = -\dot{x}\mathrm{e}^t f(\mathrm{e}^{-2t}(\dot{x}^2 + 2\dot{y}))(\dot{x}^2 + 2\dot{y})g(\mathrm{e}^{-2t}(\dot{x}^2 + 2\dot{y}))$

Table 3. Continued.

Algebras	Types	Realizations	Equations
	$\mathcal{L}^1_{4,19}$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = xp + tq$	$\ddot{x} = f(t)\dot{x},\ \ \ddot{y} = f(t)(\dot{y} - \ln\dot{x}) + g(t)$
	$\mathcal{L}^2_{4,19}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = -tl$	$\ddot{x} = f(y)\dot{y}^2,\ \ \ddot{y} = g(y)\dot{y}^2$
	$\mathcal{L}^3_{4,19}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = -tl + yp$	$\ddot{x} = f(y)\dot{y}^2 + g(y)\dot{y}^2\ln\dot{y}, \ddot{y} = g(y)\dot{y}^2$
$\mathcal{L}_{4,19}$	$\mathcal{L}^4_{4,19}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = -tl + q$	$\ddot{x} = \dot{y}^2 f(\dot{y}\mathrm{e}^{-y}), \ddot{y} = \dot{y}^2 g(\dot{y}\mathrm{e}^{-y})$
	$\mathcal{L}^5_{4,19}$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = 2tl + xp$	$\ddot{x} = t^{-3/2} f(t\dot{x}^2), \dot{x}\ddot{y} = \dot{y}t^{-3/2} f(t\dot{x}^2) + \dot{x}t^{-2} g(t\dot{x}^2)$
	$\mathcal{L}^6_{4,19}$	$X_1 = q, X_2 = p, X_3 = tp + xq, X_4 = 2tl + xp$	$\ddot{x} = t^{-3/2} f(t(\dot{x}^2 + 2\dot{y})), \ddot{y} = t^{-3/2}\dot{x} f(t(\dot{x}^2 + 2\dot{y})) + t^{-2} g(t(\dot{x}^2 + 2\dot{y}))$
	$\mathcal{L}^7_{4,19}$	$X_1 = q, X_2 = p, X_3 = l + xq, X_4 = -tl + xp$	$\ddot{x} = \dot{x}^{3/2} f\left(\frac{t\dot{x} - \dot{y}}{\dot{x}^{1/2}}\right), \ddot{y} = t\dot{x}^{3/2} f\left(\frac{t\dot{x} - \dot{y}}{\dot{x}^{1/2}}\right) + \dot{x} g\left(\frac{t\dot{x} - \dot{y}}{\dot{x}^{1/2}}\right)$
	$\mathcal{L}^{b,1}_{4,20}, b \neq 0$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = xp + (1+b)yq$	$\ddot{x} = f(t)\dot{x}, \ddot{y} = f(t)\dot{y} + g(t)\dot{x}^{(1+b)}$
	$\mathcal{L}^{b,2}_{4,20}, b \neq 0$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = btl + (1+b)xp$	$\ddot{x} = f(y)\dot{y}^{\frac{b-1}{b}}, \ddot{y} = g(y)\dot{y}^2$
	$\mathcal{L}^{0,2}_{4,20}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = xp$	$\ddot{x} = 0, \ddot{y} = g(y, \dot{y})$
	$\mathcal{L}^{b,3}_{4,20}$	$X_1 = p, X_2 = tp, X_3 = -l, X_4 = btl + (1+b)xp + q$	$\ddot{x} = \mathrm{e}^{(1-b)y} f(\dot{y}\mathrm{e}^{by}), \ddot{y} = \dot{y}^2 g(\dot{y}\mathrm{e}^{by})$
$\mathcal{L}^b_{4,20}$ $0 \leqslant \lvert b\rvert < 1$	$\mathcal{L}^{b,4}_{4,20}$	$X_1 = q, X_2 = p, X_3 = xq, X_4 = (1-b)tl + xp + (1+b)yq$	$\ddot{x} = t^{\frac{2b-1}{1-b}} f(\dot{x}t^{\frac{b}{b-1}})$ $\dot{x}\ddot{y} = \dot{y}t^{\frac{2b-1}{1-b}} f(\,) + \dot{x}t^{\frac{3b-1}{1-b}} g(\,)$
	$\mathcal{L}^{b,5}_{4,20}$	$X_1 = q, X_2 = p, X_3 = tp + xq, X_4 = (1-b)tl + xp + (1+b)yq$	$\ddot{x} = t^{\frac{2b-1}{1-b}} f(t^{\frac{2b}{b-1}}(\dot{x}^2 + 2\dot{y}))$ $\ddot{y} = \dot{x}t^{\frac{2b-1}{1-b}} f(\,) + t^{\frac{3b-1}{1-b}} g(\,)$
	$\mathcal{L}^{b,6}_{4,20}$	$X_1 = q, X_2 = p, X_3 = l + xq, X_4 = btl + xp + (1+b)yq$	$\ddot{x} = \dot{x}^{\frac{1-2b}{1-b}} f(\dot{x}^{\frac{1}{b-1}}(t\dot{x} - \dot{y}))$ $\ddot{y} = t\dot{x}^{\frac{1-2b}{1-b}} f(\,) + \dot{x} g(\,)$

Table 3. Continued.

Algebras	Types	Realizations	Equations
	$\mathcal{L}_{4,20}^{1,1}$	$X_1 = q,\, X_2 = p,\, X_3 = xq,\, X_4 = xp + 2yq$	$\ddot{x} = f(t)\dot{x},\;\; \ddot{y} = f(t)\dot{y} + g(t)\dot{x}^2$
	$\mathcal{L}_{4,20}^{1,2}$	$X_1 = p,\, X_2 = tp,\, X_3 = -l,\, X_4 = tl + 2xp$	$\ddot{x} = f(y),\, \ddot{y} = g(y)\dot{y}^2$
	$\mathcal{L}_{4,20}^{1,3}$	$X_1 = p,\, X_2 = tp,\, X_3 = -l,\, X_4 = tl + 2xp + q$	$\ddot{x} = f(\dot{y}\mathrm{e}^{y}),\, \ddot{y} = \dot{y}^2 g(\dot{y}\mathrm{e}^{y})$
$\mathcal{L}_{4,20}^{1}$	$\mathcal{L}_{4,20}^{1,4}$	$X_1 = q,\, X_2 = p,\, X_3 = tp + xq,\, X_4 = xp + 2yq$	$\ddot{x} = f(t)(\dot{x}^2 + 2\dot{y})^{1/2},\, \ddot{y} = \dot{x}(\dot{x}^2 + 2\dot{y})^{1/2} f(t) + (\dot{x}^2 + 2\dot{y})g(t)$
	$\mathcal{L}_{4,20}^{1,5}$	$X_1 = q,\, X_2 = p,\, X_3 = l + xq,\, X_4 = tl + xp + 2yq$	$\ddot{x} = \dfrac{f(\dot{x})}{t\dot{x} - \dot{y}},\, \ddot{y} = \dfrac{tf(\dot{x})}{t\dot{x} - \dot{y}} + g(\dot{x})$
	$\mathcal{L}_{4,21}^{a,1}$	$X_1 = q,\, X_2 = p,\, X_3 = l + xp,$ $X_4 = (at - x + b)l + (ax + t)p + \left(2ay + \frac{1}{2}(t^2 - x^2)\right)q$	$\ddot{x} = (1 + \dot{x}^2)^{-3/2}\mathrm{e}^{-a\arctan\dot{x}} f\left(\dfrac{t\dot{x} - \dot{y} + b}{(1 + \dot{x}^2)^{1/2}}\mathrm{e}^{-a\arctan\dot{x}}\right)$ $\ddot{y} = \dot{x} + (t + \dot{x}\dot{y} - b\dot{x})(1 + \dot{x}^2)^{1/2}\mathrm{e}^{-a\arctan\dot{x}} f(\;) + (1 + \dot{x}^2)g(\;)$
$\mathcal{L}_{4,21}^{a}$ $a \geqslant 0$	$\mathcal{L}_{4,21}^{a,2}$	$X_1 = q,\, X_2 = p,\, X_3 = -tp + xq,$ $X_4 = (1 + t^2)l + (t + a)xp + (2ay - x^2/2)q$	$\ddot{x} = (1 + t^2)^{-3/2}\mathrm{e}^{a\arctan t} f((1 + t^2)(\dot{x}^2 + 2\dot{y}))$ $\ddot{y} = -\dot{x}(1 + t^2)\,\mathrm{e}^{a\arctan t} f(\;) + (1 + \dot{x}^2)^{-2} g(\;)$ $-t(1 + t^2)^{-1}(\dot{x}^2 + 2\dot{y})$
	$\mathcal{L}_{4,22}^{1}$	$X_1 = p,\, X_2 = tp,\, X_3 = xp,\, X_4 = -(1 + t^2)l - txp$	$\ddot{x} = 0,\;\; \ddot{y} = -2(1 + t^2)^{-1}\dot{y}^2 + (1 + t^2)^{-2} g(y, (1 + t^2)\dot{y})$
	$\mathcal{L}_{4,22}^{2}$	$X_1 = l,\, X_2 = p,\, X_3 = tl + xp,\, X_4 = xl - tp$	$\ddot{x} = (1 + \dot{x}^2)\dot{y} f(y),\, \ddot{y} = (f(y)\dot{x} + g(y))\dot{y}^2$
$\mathcal{L}_{4,22}$	$\mathcal{L}_{4,22}^{3}$	$X_1 = p,\, X_2 = tp,\, X_3 = xp + yq,\, X_4 = -(1 + t^2)l - txp$	$\ddot{x} = y(1 + t^2)^{-3/2} f((1 + t^2)^{-1} y/\dot{y}),$ $\ddot{y} = -2t\dot{y}/y + (1 + t^2)^{-2} g((1 + t^2)^{-1} y/\dot{y})$
	$\mathcal{L}_{4,22}^{4}$	$X_1 = l,\, X_2 = p,\, X_3 = tl + xp,\, X_4 = xl - tp + q$	$\ddot{x} = (1 + \dot{x}^2)\dot{y} f(y + \arctan\dot{x}),$ $\ddot{y} = (\dot{x} f(y + \arctan\dot{x}) + g(y + \arctan\dot{x}))\,\dot{y}^2$
	$\mathcal{L}_{4,22}^{5}$	$X_1 = q,\, X_2 = p,\, X_3 = tl + xp + yq,\, X_4 = ctl - yp + xq$	$t\ddot{x} = \dot{x} f(v) + \dot{y} g(v),\, t\ddot{y} = \dot{y} f(v) - \dot{x} g(v),$ $v = \mathrm{e}^{c\arctan(\dot{y}/\dot{x})}\sqrt{\dot{x}^2 + \dot{y}^2}$

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